

# MUTUAL ABSOLUTE CONTINUITY OF HARMONIC AND SURFACE MEASURES FOR HÖRMANDER TYPE OPERATORS

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*Dedicated to Professor Maz'ya, on his 70th birthday*

## 1. Introduction

In this paper we study the Dirichlet problem for the sub-Laplacian associated with a system  $X = \{X_1, \dots, X_m\}$  of  $C^\infty$  real vector fields in  $\mathbb{R}^n$  satisfying Hörmander's finite rank condition

$$(1.1) \quad \text{rank Lie}[X_1, \dots, X_m] \equiv n.$$

Throughout this paper  $n \geq 3$ , and  $X_j^*$  denotes the formal adjoint of  $X_j$ . The sub-Laplacian associated with  $X$  is defined by

$$(1.2) \quad \mathcal{L}u = \sum_{j=1}^m X_j^* X_j u.$$

A distributional solution of  $\mathcal{L}u = 0$  is called  $\mathcal{L}$ -harmonic. Hörmander's hypoellipticity theorem [H] guarantees that every  $\mathcal{L}$ -harmonic function is  $C^\infty$ , hence it is a classical solution of  $\mathcal{L}u = 0$ . We consider a bounded open set  $D \subset \mathbb{R}^n$ , and study the Dirichlet problem

$$(1.3) \quad \begin{cases} \mathcal{L}u = 0 & \text{in } D, \\ u = \phi & \text{on } \partial D. \end{cases}$$

Using Bony's maximum principle [B] one can show that for any  $\phi \in C(\partial D)$  there exists a unique Perron-Wiener-Brelot solution  $H_\phi^D$  to (1.3). We focus on the boundary regularity of the solution. In particular, we identify a class of domains, which are referred to as  $ADP_X$  domains (*admissible for the Dirichlet problem*), for which we prove the mutual absolute continuity of the  $\mathcal{L}$ -harmonic measure  $d\omega^x$  and of the so-called horizontal perimeter measure  $d\sigma_X = P_X(D; \cdot)$  on  $\partial D$ . The latter constitutes the appropriate replacement for the standard surface measure on  $\partial D$  and plays a central role in sub-Riemannian geometry. Moreover, we show that a reverse Hölder inequality holds for a suitable Poisson kernel which is naturally associated with the system  $X$ . As a consequence of such reverse Hölder inequality we then derive the solvability of (1.3) for boundary data  $\phi \in L^p(\partial D, d\sigma_X)$ , for  $1 < p \leq \infty$ . If instead the domain  $D$  belongs to the smaller class  $\sigma - ADP_X$  introduced in Definition 8.10 below, we prove that  $\mathcal{L}$ -harmonic measure is mutually absolutely continuous with respect to the standard surface measure, and we are able to solve the Dirichlet problem for (1.2) for boundary data  $\phi \in L^p(\partial D, d\sigma)$ , for  $1 < p \leq \infty$ .

The connection between harmonic and surface measure is a central question in the study of boundary value problems for second order partial differential equations. As it is well-known a basic result of Brelot allows to solve the Dirichlet problem for the standard Laplacian when the boundary datum is in  $L^1$  with respect to the harmonic measure. However, since the latter is difficult to pin down, it becomes important to know for what domains one can solve the Dirichlet problem when the boundary data are in some  $L^p$  space with respect to the ordinary

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surface measure  $d\sigma$ . In his ground-breaking 1977 paper [Da1] Dahlberg was able to settle the long standing conjecture that in a Lipschitz domain in  $\mathbb{R}^n$  harmonic measure for the Laplacian and Hausdorff measure  $H^{n-1}$  restricted to the boundary are mutually absolutely continuous. One should also see the sequel paper [Da2] where the mutual absolute continuity was obtained as a consequence of the reverse Hölder inequality for the kernel function  $k = d\omega/d\sigma$ . For  $C^1$  domains Dahlberg's result was also independently proved by Fabes, Jodeit and Rivière [FJR] by the method of layer potentials.

The results in this paper should be considered as a subelliptic counterpart of Dahlberg's results in [Da1], [Da2]. There are however four aspects which substantially differ from the analysis of the ordinary Laplacian, and they are all connected with the presence of the so-called characteristic points on the boundary. In order to describe these aspects we recall that given a  $C^1$  domain  $D \subset \mathbb{R}^n$ , a point  $x_o \in \partial D$  is called *characteristic* for the system  $X = \{X_1, \dots, X_m\}$  if indicating with  $\mathbf{N}(x_o)$  a normal vector to  $\partial D$  in  $x_o$  one has

$$\langle \mathbf{N}(x_o), X_1(x_o) \rangle = \dots = \langle \mathbf{N}(x_o), X_m(x_o) \rangle = 0.$$

The characteristic set of  $D$ , hereafter denoted by  $\Sigma = \Sigma_{D,X}$ , is the collection of all characteristic points of  $\partial D$ . It is a closed subset of  $\partial D$ , and it is compact if  $D$  is bounded. We next introduce the most important prototype of a sub-Riemannian space: the Heisenberg group  $\mathbb{H}^n$ . This is the stratified nilpotent Lie group of step two whose underlying manifold is  $\mathbb{C}^n \times \mathbb{R}$  with group law  $(z, t) \circ (z', t') = (z + z', t + t' - \frac{1}{2} \operatorname{Im}(z \cdot \bar{z}'))$ . If  $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ , and we identify  $z = x + iy \in \mathbb{C}^n$  with the vector  $(x, y) \in \mathbb{R}^{2n}$ , then in the real coordinates  $(x, y, t) \in \mathbb{R}^{2n+1}$  a basis for the Lie algebra of left-invariant vector fields on  $\mathbb{H}^n$  is given by the vector fields

$$(1.4) \quad X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial t}, \quad j = 1, \dots, n, \quad \frac{\partial}{\partial t}.$$

In view of the commutation relations

$$[X_j, X_{n+k}] = \delta_{jk} \frac{\partial}{\partial t}, \quad j, k = 1, \dots, n,$$

the system  $X = \{X_1, \dots, X_{2n}\}$  generates the Lie algebra of  $\mathbb{H}^n$ . The real part of the Kohn-Spencer sub-Laplacian on  $\mathbb{H}^n$  is given by

$$(1.5) \quad \mathcal{L}_o = \sum_{j=1}^{2n} X_j^2 = \Delta_z + \frac{|z|^2}{4} D_{tt} + D_t \left( \sum_{j=1}^n x_j D_{y_j} - y_j D_{x_j} \right).$$

This remarkable operator plays an ubiquitous role in several branches of mathematics and of the applied sciences. We stress that  $\mathcal{L}_o$  fails to be elliptic at every point. Concerning the distinctions mentioned above we note:

1) Differently from the classical case, in the subelliptic Dirichlet problem (1.3) the Euclidean smoothness of the ground domain is of no significance from the standpoint of the intrinsic geometry near the characteristic set  $\Sigma$ . In this geometry even a domain with real analytic boundary looks like a cuspidal domain near one of its characteristic points. Since bounded domains typically have non-empty characteristic set it follows that the notion of “Lipschitz domain” is not as important as in the Euclidean setting, and one has to abandon it in favor of a more general one based on purely metrical properties, see [CG1]. With these comments in mind, in this paper we will assume that the domain  $D$  in (1.3) be  $NTA_X$  (*non-tangentially accessible* with respect to the Carnot-Carathéodory distance associated with the system  $X$ , see Definition 8.1 below) and  $C^\infty$ . The former property allows us to use some fundamental results developed in [CG1], whereas the smoothness assumption permits to use tools from calculus away from the characteristic set. In this connection we mention that the  $C^\infty$  hypothesis guarantees, in view of

the results in [KN1], that the Green function for (1.3) and singularity at a given point in  $D$  is smooth up to the boundary away from  $\Sigma$ , see Theorem 3.12 below.

2) Another striking phenomenon is that in the subelliptic Dirichlet problem nonnegative  $\mathcal{L}$ -harmonic functions which vanish on a portion of the boundary can do so at very different rates. The dual aspect of this phenomenon is that nonnegative  $\mathcal{L}$ -harmonic functions which blow up at the boundary (such as for instance the Poisson kernel) have very different rates of blow-up depending on whether the limit point is characteristic or not, see [GV]. This is in sharp contrast with the classical setting. It is well-known [G] that in a  $C^{1,1}$  domain all nonnegative harmonic functions (or solutions to more general elliptic and parabolic equations) vanishing on a portion of the boundary must vanish exactly like the distance to the boundary itself. This fails miserably in the subelliptic setting because of characteristic points on the boundary. For instance, in  $\mathbb{H}^n$  the so-called gauge ball

$$B = \{(z, t) \in \mathbb{H}^n \mid |z|^4 + 16t^2 < 1\}$$

is a real analytic domain with two isolated characteristic points  $P^\pm = (0, \pm \frac{1}{4})$ . With  $\mathcal{L}_o$  defined by (1.5), the function  $u(z, t) = t + \frac{1}{4}$  is a nonnegative  $\mathcal{L}_o$ -harmonic function in  $B$  which along the  $t$ -axis vanishes at the (characteristic) boundary point  $P^- = (0, -\frac{1}{4})$  as the square of the Carnot-Carathéodory distance to  $P^-$ . On the other hand, the function  $u(z, t) = x_1 + 1$  is a nonnegative  $\mathcal{L}_o$ -harmonic function in  $B$  which along the  $x_1$ -axis vanishes at the (non-characteristic) boundary point  $P_1 = (-e_1, 0)$ , where  $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{2n}$ , like the distance to  $P_1$ . Thus, there is not one single rate of vanishing for nonnegative  $\mathcal{L}_o$ -harmonic functions in smooth domains in  $\mathbb{H}^n$ ! Despite this negative phenomenon in [CG1] two of us proved that in a  $NTA_X$  domain all nonnegative  $\mathcal{L}$ -harmonic functions vanishing on a portion of the boundary (characteristic or not) must do so at the same rate. This result, known as the *comparison theorem*, plays a fundamental role in the present paper. Returning to the above example of the gauge ball  $B \subset \mathbb{H}^n$ , the comparison theorem implies in particular that all nonnegative solutions of  $\mathcal{L}_o u = 0$  which vanish in a boundary neighborhood of the point  $P^- = (0, -\frac{1}{4})$ , must vanish non-tangentially like the square of the distance to the boundary (and not linearly like in the classical case)!

3) The third aspect which we want to emphasize is closely connected with the discussion in 1) and leads us to introduce the third main assumption in the present paper. In [J1], [J2] D. Jerison studied the Dirichlet problem (1.3) near characteristic points for  $\mathcal{L}_o$ . He proved in [J1] that for a  $C^\infty$  domain  $D \subset \mathbb{H}^n$  if the datum  $\phi$  belongs to a Folland-Stein Hölder class  $\Gamma^\beta$ , then  $H_\phi^D$  is in  $\Gamma^\alpha(\overline{D})$ , for some  $\alpha$  depending on  $\beta$  and on the domain  $D$ . It was also shown in [J1] that, given any  $\alpha \in (0, 1)$  there exists  $M = M(\alpha) > 0$  for which the real analytic domain

$$\Omega_M = \{(z, t) \in \mathbb{H}^n \mid t > -M|z|^2\},$$

admits a  $\mathcal{L}_o$ -harmonic function  $u$  such that  $u = 0$  on  $\partial\Omega_M$  and which belongs exactly to the Hölder class  $\Gamma^\alpha$  (in the sense that it is not any smoother) in any neighborhood of the (characteristic) boundary point  $e = (0, 0)$ . Once again, this example shows that, despite the (Euclidean) smoothness of the domain and of the boundary datum, near a characteristic point the domain appears quite non-smooth with respect to the intrinsic geometry of the vector fields  $X_1, \dots, X_{2n}$ . In fact, since the paraboloid  $\Omega$  is a scale invariant region with respect to the non-isotropic group dilations  $(z, t) \rightarrow (\lambda z, \lambda^2 t)$ , the smooth domain  $\Omega_M$  should be thought of as a non-convex cone from the point of view of the intrinsic geometry of  $\mathcal{L}_o$  (for a discussion of Jerison's example see section 4). This suggests that by imposing a condition similar to the classical Poincaré tangent outer sphere [P] one should be able to rule out Jerison's negative example and possibly control the intrinsic gradient  $XG$  of the Green function near the characteristic set. This intuition was proved successful in the papers [LU1], [CGN1], which were respectively concerned with the Heisenberg group and with Carnot groups of Heisenberg type. In this paper we generalize this idea and prove the boundedness of the Poisson kernel in a neighborhood of the boundary under the hypothesis that the domain  $D$  in (1.3) satisfy what we call a *tangent outer  $X$ -ball condition*.

It is worth emphasizing that the  $X$ -balls in our definition are not metric balls, but instead they are the (smooth) level sets of the fundamental solution of the sub-Laplacian  $\mathcal{L}$ . The metric balls are not smooth (see [CG1]) and therefore it would not be possible to have a notion of tangency based on these sets.

4) In Dahlberg's mentioned theorem on the mutual absolute continuity between harmonic and surface measure in a Lipschitz domain  $D \subset \mathbb{R}^n$  there is one important property which, although confined to the background, plays a central role. If we denote by  $\sigma = H^{n-1}|_{\partial D}$  the surface measure on the boundary, then there exists constants  $\alpha, \beta > 0$  depending on  $n$  and on the Lipschitz character of  $D$  such that

$$\alpha r^{n-1} \leq \sigma(\partial D \cap B(x, r)) \leq \beta r^{n-1},$$

for any  $x \in \partial D$  and any  $r > 0$ . A property like this is referred to as the 1-Ahlfors regularity of  $\sigma$ , and thanks to it surface measure is the natural measure on  $\partial D$ . Things are quite different in the subelliptic Dirichlet problem. Consider in fact the gauge ball  $B$  as in 2), with its two (isolated) characteristic points  $P^\pm = (0, \pm \frac{1}{4})$  of  $\partial B$ . Simple calculations show that denoting by  $B(P^\pm, r)$  a gauge ball centered at one of the points  $P^\pm$  with radius  $r$ , then one has for small  $r > 0$

$$(1.6) \quad \sigma(\partial B \cap B(P^\pm, r)) \cong r^{Q-2},$$

where  $Q = 2n + 2$  is the so-called homogeneous dimension of  $\mathbb{H}^n$  relative to the non-isotropic dilations  $(z, t) \rightarrow (\lambda z, \lambda^2 t)$  associated with the grading of the Lie algebra of  $\mathbb{H}^n$ . The latter equation shows that at the characteristic points  $P^\pm$  surface measure becomes quite singular and it does not scale correctly with respect to the non-isotropic group dilations. The appropriate "surface measure" in sub-Riemannian geometry is instead the so-called horizontal perimeter  $P_X(D; \cdot)$  introduced in [CDG2] which on surface metric balls is defined in the following way

$$\sigma_X(\partial D \cap B_d(x, r)) \stackrel{\text{def}}{=} P_X(D; B_d(x, r)).$$

To motivate such appropriateness we recall that it was proved in [DGN1], [DGN2] that for every  $C^2$  bounded domain  $D \subset \mathbb{H}^n$  one has for every  $x \in \partial D$  and every  $0 < r < R_o(D)$

$$\alpha r^{Q-1} \leq \sigma_X(\partial D \cap B_d(x, r)) \leq \beta r^{Q-1}.$$

Now it was also shown in these papers that the inequality in the right-hand side alone suffices to establish the existence of the traces of Sobolev functions on the boundary. Remarkably, as we prove in Theorem 1.3 below, such a one-sided Ahlfors property also suffices to establish the mutual absolute continuity of  $\mathcal{L}$ -harmonic and horizontal perimeter measure. Such property will constitute the last basic assumption of our results, to which we finally turn. We need to introduce the relevant class of domains.

**Definition 1.1.** *Given a system  $X = \{X_1, \dots, X_m\}$  of smooth vector fields satisfying (1.1), we say that a connected bounded open set  $D \subset \mathbb{R}^n$  is admissible for the Dirichlet problem (1.3) with respect to the system  $X$ , or simply  $ADP_X$ , if:*

- i)  $D$  is of class  $C^\infty$ ;
- ii)  $D$  is non-tangentially accessible ( $NTA_X$ ) with respect to the Carnot-Carathéodory metric associated to the system  $\{X_1, \dots, X_m\}$  (see Definition 8.1);
- iii)  $D$  satisfies a uniform tangent outer  $X$ -ball condition (see Definition 6.2);
- iv) The horizontal perimeter measure is upper 1-Ahlfors regular. This means that there exist  $A, R_o > 0$  depending on  $X$  and  $D$  such that for every  $x \in \partial D$  and  $0 < r < R_o$  one has

$$\sigma_X(\partial D \cap B_d(x, r)) \leq A \frac{|B_d(x, r)|}{r}.$$

The constants appearing in iv) and in Definitions 6.2 and 8.1 will be referred to as the  $ADP_X$ -parameters of  $D$ . We introduce next a central character in this play, the subelliptic

Poisson kernel of  $D$ . In fact, we define two such functions, each one playing a different role. Let  $G(x, y) = G_D(x, y) = G(y, x)$  indicate the Green function for the sub-Laplacian (1.2) and for an  $ADP_X$  domain  $D$ <sup>1</sup>. By Hörmander's theorem [H] and the results in [KN1], see Theorem 3.12 below, for any fixed  $x \in D$  the function  $y \rightarrow G(x, y)$  is  $C^\infty$  up to the boundary in a suitably small neighborhood of any non-characteristic point  $y_o \in \partial D$ . Let  $\nu(y)$  indicate the outer unit normal in  $y \in \partial D$ . At every point  $y \in \partial D$  we denote by  $\mathbf{N}^X(y)$  the vector defined by

$$\mathbf{N}^X(y) = (\langle \nu(y), X_1(y) \rangle, \dots, \langle \nu(y), X_m(y) \rangle) .$$

We also set

$$W(y) = |\mathbf{N}^X(y)| = \sqrt{\langle \nu(y), X_1(y) \rangle^2 + \dots + \langle \nu(y), X_m(y) \rangle^2} .$$

We note explicitly that it was proved in [CDG2] that on  $\partial D$

$$d\sigma_X = W d\sigma .$$

Denoting with  $\Sigma$  the characteristic set of  $D$ , we remark that the vector  $\mathbf{N}^X(y) = 0$  if and only if  $y \in \Sigma$ . For  $y \in \partial D \setminus \Sigma$  we define the *horizontal Gauss map* at  $y$  by letting

$$\nu^X(y) = \frac{\mathbf{N}^X(y)}{|\mathbf{N}^X(y)|} .$$

**Definition 1.2.** *Given a  $C^\infty$  bounded open set  $D \subset \mathbb{R}^n$ , for every  $(x, y) \in D \times (\partial D \setminus \Sigma)$  we define the subelliptic Poisson kernels as follows*

$$P(x, y) = \langle XG(x, y), \mathbf{N}^X(y) \rangle , \quad K(x, y) = \frac{P(x, y)}{W(y)} = \langle XG(x, y), \nu^X(y) \rangle .$$

We emphasize here that the reason for which in the definition of  $P(x, y)$  and  $K(x, y)$  we restrict  $y$  to  $\partial D \setminus \Sigma$  is that, as we have explained in 3) above (see also section 4), the horizontal gradient  $XG(x, y)$  may not be defined at points of  $\Sigma$ . Since as we have observed the function  $W$  vanishes on  $\Sigma$ , it should be clear that the function  $K(x, y)$  is more singular than  $P(x, y)$  at the characteristic points. However, such additional singularity is balanced by the fact that the density  $W$  of the measure  $\sigma_X$  with respect to surface measure vanishes at the characteristic points. As a consequence,  $K(x, y)$  is the appropriate subelliptic Poisson kernel with respect to the intrinsic measure  $\sigma_X$ , whereas  $P(x, y)$  is more naturally attached to the “wrong measure”  $\sigma$ .

Hereafter, for  $x \in \partial D$  it will be convenient to indicate with  $\Delta(x, r) = \partial D \cap B_d(x, r)$ , the boundary metric ball centered at  $x$  with radius  $r > 0$ . The first main result in this paper is contained the following theorem.

**Theorem 1.3.** *Let  $D \subset \mathbb{R}^n$  be a  $ADP_X$  domain. For every  $p > 1$  and any fixed  $x_1 \in D$  there exist positive constants  $C, R_1$ , depending on  $p, M, R_o, x_1$ , and on the  $ADP_X$  parameters, such that for  $x_o \in \partial D$  and  $0 < r < R_1$  one has*

$$\left( \frac{1}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} K(x_1, y)^p d\sigma_X(y) \right)^{\frac{1}{p}} \leq C \frac{1}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} K(x_1, y) d\sigma_X(y) .$$

Moreover, the measures  $d\omega^{x_1}$  and  $d\sigma_X$  are mutually absolutely continuous.

By combining Theorem 1.3 with the results in [CG1] we can solve the Dirichlet problem for boundary data in  $L^p$  with respect to the perimeter measure  $d\sigma_X$ . To state the relevant results we need to introduce a definition. Given  $D$  as in Theorem 1.3, for any  $y \in \partial D$  and  $\alpha > 0$  a nontangential region at  $y$  is defined by

$$\Gamma_\alpha(y) = \{x \in D \mid d(x, y) \leq (1 + \alpha)d(x, \partial D)\} .$$

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<sup>1</sup>In [B] it was proved that any bounded open set admits a Green function

Given a function  $u \in C(D)$ , the  $\alpha$ -nontangential maximal function of  $u$  at  $y$  is defined by

$$N_\alpha(u)(y) = \sup_{x \in \Gamma_\alpha(y)} |u(x)| .$$

**Theorem 1.4.** *Let  $D \subset \mathbb{R}^n$  be a  $ADP_X$  domain. For every  $p > 1$  there exists a constant  $C > 0$  depending on  $D, X$  and  $p$  such that if  $f \in L^p(\partial D, d\sigma_X)$ , then*

$$H_f^D(x) = \int_{\partial D} f(y) K(x, y) d\sigma_X(y) ,$$

and

$$\|N_\alpha(H_f^D)\|_{L^p(\partial D, d\sigma_X)} \leq C \|f\|_{L^p(\partial D, d\sigma_X)} .$$

Furthermore,  $H_f^D$  converges nontangentially  $\sigma_X$ -a.e. to  $f$  on  $\partial D$ .

Theorems 1.3 and 1.4 constitute appropriate sub-elliptic versions of Dahlberg's mentioned results in [Da1], [Da2]. These theorems generalize those in [CGN2] relative to Carnot groups of Heisenberg type. We mention at this point that, as we prove in Theorem 8.3 below, for any  $C^{1,1}$  domain  $D \subset \mathbb{R}^n$  which is  $NTA_X$  the horizontal perimeter measure is lower 1-Ahlfors (this is a basic consequence of the isoperimetric inequality in [GN1]). Combining this result with the assumption *iv*) in Definition 1.1, we conclude that for any  $ADP_X$  domain the measure  $\sigma_X$  is 1-Ahlfors. In particular,  $\sigma_X$  is also doubling, see Corollary 8.4. This information plays a crucial role in the proof of Theorem 1.4.

On the other hand, even if the ordinary surface measure  $\sigma$  is the “wrong one” in the subelliptic Dirichlet problem, it would still be highly desirable to know if there exist situations in which (1.3) can be solved for boundary data in some  $L^p$  with respect to  $d\sigma$ . To address this question in Definition 8.10 we introduce the class of  $\sigma - ADP_X$  domains. The latter differs from that of  $ADP_X$  domains for the fact that the assumption *iv*) is replaced by the following *balanced-degeneracy* assumption on  $\sigma$ : there exist  $B, R_o > 0$  depending on  $X$  and  $D$  such that for every  $x_o \in \partial D$  and  $0 < r < R_o$  one has

$$\left( \max_{y \in \Delta(x_o, r)} W(y) \right) \sigma(\Delta(x_o, r)) \leq B \frac{|B_d(x_o, r)|}{r} .$$

As we have previously observed surface measure becomes singular near a characteristic point. On the other hand, the angle function  $W$  vanishes, thus balancing the singularities of  $\sigma$ . For  $\sigma - ADP_X$ -domains we obtain the following two results which respectively establish the mutual absolute continuity of  $\mathcal{L}$ -harmonic and surface measure  $d\sigma$ , and the solvability of the Dirichlet problem with data in  $L^p(\partial D, d\sigma)$ .

**Theorem 1.5.** *Let  $D \subset \mathbb{R}^n$  be a  $\sigma - ADP_X$  domain. Fix  $x_1 \in D$ . For every  $p > 1$  there exist positive constants  $C, R_1$ , depending on  $p, M, R_o, x_1$ , and on the  $\sigma - ADP_X$  parameters, such that for every  $y \in \partial D$  and  $0 < r < R_1$  one has*

$$\left( \frac{1}{\sigma(\Delta(x_o, r))} \int_{\Delta(x_o, r)} P(x_1, y)^p d\sigma(y) \right)^{\frac{1}{p}} \leq C \frac{1}{\sigma(\Delta(x_o, r))} \int_{\Delta(x_o, r)} P(x_1, y) d\sigma(y) .$$

Moreover, the measures  $d\omega^x$  and  $d\sigma$  are mutually absolutely continuous.

We mention explicitly that a basic consequence of Theorem 1.5 is that the standard surface measure on the boundary of a  $\sigma - ADP_X$  domain is doubling.

**Theorem 1.6.** *Let  $D \subset \mathbb{R}^n$  be a  $\sigma - ADP_X$  domain. For every  $p > 1$  there exists a constant  $C > 0$  depending on  $D, X$  and  $p$  such that if  $f \in L^p(\partial D, d\sigma)$ , then*

$$H_f^D(x) = \int_{\partial D} f(y) P(x, y) d\sigma(y) ,$$

and

$$\|N_\alpha(H_f^D)\|_{L^p(\partial D, d\sigma)} \leq C \|f\|_{L^p(\partial D, d\sigma)}.$$

Furthermore,  $H_f^D$  converges nontangentially  $\sigma$ -a.e. to  $f$  on  $\partial D$ .

Concerning Theorems 1.3, 1.4, 1.5 and 1.6 we mention that large classes of domains to which they apply were found in [CGN2], but one should also see [LU1] for domains satisfying assumption *iii*) in Definition 1.1. The discussion of these examples is taken up in section 9.

In closing we briefly describe the organization of the paper. In section 2 we collect some known results on Carnot-Carathéodory metrics which are needed in the paper. In section 3 we discuss some known results on the subelliptic Dirichlet problem which constitute the potential theoretic backbone of the paper. In section 4 we discuss Jerison's mentioned example.

Section 5 is devoted to proving some new interior a priori estimates of Cauchy-Schauder type. Such estimates are obtained by means of a family of subelliptic mollifiers which were introduced by Danielli and two of us in [CDG1], see also [CDG2]. The main results are Theorems 5.1, 5.5, and Corollary 5.3. We feel that, besides being instrumental to the present paper, these results will prove quite useful in future research on the subject.

In section 6 we use the interior estimates in Theorem 5.1 to prove that if a domain satisfies a uniform outer tangent  $X$ -ball condition, then the horizontal gradient of the Green function  $G$  is bounded up to the boundary, hence, in particular, near  $\Sigma$ , see Theorem 6.6. The proof of such result rests in an essential way on the linear growth estimate provided by Theorem 6.3. Another crucial ingredient is Lemma 6.1 which allows a delicate control of some ad-hoc subelliptic barriers. In the final part of the section we show that, by requesting the uniform outer  $X$ -ball condition only in a neighborhood of the characteristic set  $\Sigma$ , we are still able to obtain the boundedness of the horizontal gradient of  $G$  up to the characteristic set, although we now loose the uniformity in the estimates, see Theorem 6.9, 6.10 and Corollary 6.11.

In section 7 we establish a Poisson type representation formula for domains which satisfy the uniform outer  $X$ -ball condition in a neighborhood of the characteristic set. This result generalizes a similar Poisson type formula in the Heisenberg group  $\mathbb{H}^n$  obtained by Lanconelli and Uguzzoni in [LU1], and extended in [CGN2] to Carnot groups of Heisenberg type. If generically the Green function of a smooth domain had bounded horizontal gradient *up to the characteristic set*, then such Poisson formula would follow in an elementary way from integration by parts. As we previously stressed, however, things are not so simple and the boundedness of  $XG$  fails in general near the characteristic set. However, when  $D \subset \mathbb{R}^n$  satisfies the uniform outer  $X$ -ball condition in a neighborhood of the characteristic set, then combining Theorem 6.6 with the estimate

$$K(x, y) \leq |XG(x, y)|, \quad x \in D, y \in \partial D,$$

see (7.7), we prove the boundedness of the Poisson kernel  $y \rightarrow K(x, y)$  on  $\partial D$ . The main result in section 7 is Theorem 7.10. This representation formula with the estimates of the Green function in sections 5 and 6 lead to a priori estimates in  $L^p$  for the solution to (1.3) when the datum  $\phi \in C(\partial D)$ . Solvability of (1.3) with data in Lebesgue classes requires, however, a much deeper analysis.

The first observation is that the outer ball condition alone does not guarantee the development of a rich potential theory. For instance, it may not be possible to find: a) Good nontangential regions of approach to the boundary from within the domain; b) Appropriate interior Harnack chains of nontangential balls. This is where the basic results on  $NTA_X$  domains from [CG1] enter the picture. In the opening of section 8 we recall the definition of  $NTA_X$ -domain along with those results from [CG1] which constitute the foundations of the present study. Using these results we establish Theorem 8.9. The remaining part of the section is devoted to proving Theorems 1.3, 1.4, 1.5 and 1.6.

Finally, section 9 is devoted to the discussion of examples of  $ADP_X$  and  $\sigma - ADP_X$  domains and of some open problems.

## 2. Preliminaries

In  $\mathbb{R}^n$ , with  $n \geq 3$ , we consider a system  $X = \{X_1, \dots, X_m\}$  of  $C^\infty$  vector fields satisfying Hörmander's finite rank condition (1.1). A piecewise  $C^1$  curve  $\gamma : [0, T] \rightarrow \mathbb{R}^n$  is called *sub-unitary* [FP] if whenever  $\gamma'(t)$  exists one has for every  $\xi \in \mathbb{R}^n$

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^m \langle X_j(\gamma(t)), \xi \rangle^2.$$

We note explicitly that the above inequality forces  $\gamma'(t)$  to belong to the span of  $\{X_1(\gamma(t)), \dots, X_m(\gamma(t))\}$ . The sub-unit length of  $\gamma$  is by definition  $l_s(\gamma) = T$ . Given  $x, y \in \mathbb{R}^n$ , denote by  $\mathcal{S}_\Omega(x, y)$  the collection of all sub-unitary  $\gamma : [0, T] \rightarrow \Omega$  which join  $x$  to  $y$ . The accessibility theorem of Chow and Rashevsky, [Ra], [Ch], states that, given a connected open set  $\Omega \subset \mathbb{R}^n$ , for every  $x, y \in \Omega$  there exists  $\gamma \in \mathcal{S}_\Omega(x, y)$ . As a consequence, if we pose

$$d_\Omega(x, y) = \inf \{l_s(\gamma) \mid \gamma \in \mathcal{S}_\Omega(x, y)\},$$

we obtain a distance on  $\Omega$ , called the *Carnot-Carathéodory distance on  $\Omega$* , associated with the system  $X$ . When  $\Omega = \mathbb{R}^n$ , we write  $d(x, y)$  instead of  $d_{\mathbb{R}^n}(x, y)$ . It is clear that  $d(x, y) \leq d_\Omega(x, y)$ ,  $x, y \in \Omega$ , for every connected open set  $\Omega \subset \mathbb{R}^n$ . In [NSW] it was proved that for every connected  $\Omega \subset \subset \mathbb{R}^n$  there exist  $C, \epsilon > 0$  such that

$$(2.1) \quad C |x - y| \leq d_\Omega(x, y) \leq C^{-1} |x - y|^\epsilon, \quad x, y \in \Omega.$$

This gives  $d(x, y) \leq C^{-1} |x - y|^\epsilon$ ,  $x, y \in \Omega$ , and therefore

$$i : (\mathbb{R}^n, |\cdot|) \rightarrow (\mathbb{R}^n, d) \quad \text{is continuous.}$$

It is easy to see that also the continuity of the opposite inclusion holds [GN1], hence the metric and the Euclidean topology are compatible.

For  $x \in \mathbb{R}^n$  and  $r > 0$ , we let  $B_d(x, r) = \{y \in \mathbb{R}^n \mid d(x, y) < r\}$ . The basic properties of these balls were established by Nagel, Stein and Wainger in their seminal paper [NSW]. Denote by  $Y_1, \dots, Y_l$  the collection of the  $X_j$ 's and of those commutators which are needed to generate  $\mathbb{R}^n$ . A formal "degree" is assigned to each  $Y_i$ , namely the corresponding order of the commutator. If  $I = (i_1, \dots, i_n)$ ,  $1 \leq i_j \leq l$  is a  $n$ -tuple of integers, following [NSW] we let  $d(I) = \sum_{j=1}^n \deg(Y_{i_j})$ , and  $a_I(x) = \det(Y_{i_1}, \dots, Y_{i_n})$ . The *Nagel-Stein-Wainger polynomial* is defined by

$$(2.2) \quad \Lambda(x, r) = \sum_I |a_I(x)| r^{d(I)}, \quad r > 0.$$

For a given bounded open set  $U \subset \mathbb{R}^n$ , we let

$$(2.3) \quad Q = \sup \{d(I) \mid |a_I(x)| \neq 0, x \in U\}, \quad Q(x) = \inf \{d(I) \mid |a_I(x)| \neq 0\},$$

and notice that  $n \leq Q(x) \leq Q$ . The numbers  $Q$  and  $Q(x)$  are respectively called the *local homogeneous dimension* of  $U$  and the homogeneous dimension at  $x$  with respect to the system  $X$ .

**Theorem 2.1** ([NSW]). *For every bounded set  $U \subset \mathbb{R}^n$ , there exist constants  $C, R_o > 0$  such that, for any  $x \in U$ , and  $0 < r \leq R_o$ ,*

$$(2.4) \quad C \Lambda(x, r) \leq |B_d(x, r)| \leq C^{-1} \Lambda(x, r).$$



As a consequence, one has with  $C_1 = 2^Q$

$$(2.5) \quad |B_d(x, 2r)| \leq C_1 |B_d(x, r)| \quad \text{for every } x \in U \quad \text{and} \quad 0 < r \leq R_o.$$

The numbers  $C_1, R_o$  in (2.5) will be referred to as the *characteristic local parameters* of  $U$ . Because of (2.2), if we let

$$(2.6) \quad E(x, r) = \frac{\Lambda(x, r)}{r^2},$$

then the function  $r \rightarrow E(x, r)$  is strictly increasing. We denote by  $F(x, \cdot)$  the inverse function of  $E(x, \cdot)$ , so that  $F(x, E(x, r)) = r$ . Let  $\Gamma(x, y) = \Gamma(y, x)$  be the positive fundamental solution of the sub-Laplacian

$$\mathcal{L} = \sum_{j=1}^m X_j^* X_j,$$

and consider its level sets

$$\Omega(x, r) = \left\{ y \in \mathbb{R}^n \mid \Gamma(x, y) > \frac{1}{r} \right\}.$$

The following definition plays a key role in this paper.

**Definition 2.2.** For every  $x \in \mathbb{R}^n$ , and  $r > 0$ , the set

$$B(x, r) = \left\{ y \in \mathbb{R}^n \mid \Gamma(x, y) > \frac{1}{E(x, r)} \right\}$$

will be called the  $X$ -ball, centered at  $x$  with radius  $r$ .

We note explicitly that

$$B(x, r) = \Omega(x, E(x, r)), \quad \text{and that} \quad \Omega(x, r) = B(x, F(x, r)).$$

One of the main geometric properties of the  $X$ -balls, is that they are equivalent to the Carnot-Carathéodory balls. To see this, we recall the following important result, established independently in [NSW], [SC]. Hereafter, the notation  $Xu = (X_1 u, \dots, X_m u)$  indicates the sub-gradient of a function  $u$ , whereas  $|Xu| = (\sum_{j=1}^m (X_j u)^2)^{\frac{1}{2}}$  will denote its length.

**Theorem 2.3.** Given a bounded set  $U \subset \mathbb{R}^n$ , there exists  $R_o$ , depending on  $U$  and on  $X$ , such that for  $x \in U$ ,  $0 < d(x, y) \leq R_o$ , one has for  $s \in \mathbb{N} \cup \{0\}$ , and for some constant  $C = C(U, X, s) > 0$

$$(2.7) \quad |X_{j_1} X_{j_2} \dots X_{j_s} \Gamma(x, y)| \leq C^{-1} \frac{d(x, y)^{2-s}}{|B_d(x, d(x, y))|},$$

$$\Gamma(x, y) \geq C \frac{d(x, y)^2}{|B_d(x, d(x, y))|}.$$

In the first inequality in (2.7), one has  $j_i \in \{1, \dots, m\}$  for  $i = 1, \dots, s$ , and  $X_{j_i}$  is allowed to act on either  $x$  or  $y$ .

In view of (2.5), (2.7), it is now easy to recognize that, given a bounded set  $U \subset \mathbb{R}^n$ , there exists  $a > 1$ , depending on  $U$  and  $X$ , such that

$$(2.8) \quad B_d(x, a^{-1}r) \subset B(x, r) \subset B_d(x, ar),$$

for  $x \in U, 0 < r \leq R_o$ . We observe that, as a consequence of (2.4), and of (2.7), one has

$$(2.9) \quad C d(x, y) \leq F\left(x, \frac{1}{\Gamma(x, y)}\right) \leq C^{-1} d(x, y),$$

for all  $x \in U, 0 < d(x, y) \leq R_o$ .

We observe that for a Carnot group  $\mathbb{G}$  of step  $k$ , if  $\mathfrak{g} = V_1 \oplus \dots \oplus V_k$  is a stratification of the Lie algebra of  $\mathbb{G}$ , then one has  $\Lambda(x, r) = \text{const } r^Q$ , for every  $x \in \mathbb{G}$  and every  $r > 0$ , with  $Q = \sum_{j=1}^k j \dim V_j$ , the homogeneous dimension of the group  $\mathbb{G}$ . In this case  $Q(x) \equiv Q$ .

In the sequel the following properties of a Carnot-Carathéodory space will be useful.

**Proposition 2.4.** *( $\mathbb{R}^n, d$ ) is locally compact. Furthermore, for any bounded set  $U \subset \mathbb{R}^n$  there exists  $R_o = R_o(U) > 0$  such that the closed balls  $\bar{B}(x_o, R)$ , with  $x_o \in U$  and  $0 < R < R_o$ , are compact.*

**Remark 2.5.** *Compactness of balls of large radii may fail in general, see [GN1]. However, there are important cases in which Proposition 2.4 holds globally, in the sense that one can take  $U$  to coincide with the whole ambient space and  $R_o = \infty$ . One example is that of Carnot groups. Another interesting case is that when the vector fields  $X_j$  have coefficients which are globally Lipschitz, see [GN1], [GN2]. Henceforth, for any given bounded set  $U \subset \mathbb{R}^n$  we will always assume that the local parameter  $R_o$  has been chosen so to accommodate Proposition 2.4.*

### 3. The Dirichlet problem

In what follows, given a system  $X = \{X_1, \dots, X_m\}$  of  $C^\infty$  vector fields in  $\mathbb{R}^n$  satisfying (1.1), and an open set  $D \subset \mathbb{R}^n$ , for  $1 \leq p \leq \infty$  we denote by  $\mathcal{L}^{1,p}(D)$  the Banach space  $\{f \in L^p(D) \mid X_j f \in L^p(D), j = 1, \dots, m\}$  endowed with its natural norm

$$\|f\|_{\mathcal{L}^{1,p}(D)} = \|f\|_{L^p(D)} + \sum_{j=1}^m \|X_j f\|_{L^p(D)}.$$

The local space  $\mathcal{L}_{loc}^{1,p}(D)$  has the usual meaning, whereas for  $1 \leq p < \infty$  the space  $\mathcal{L}_0^{1,p}(D)$  is defined as the closure of  $C_0^\infty(D)$  in the norm of  $\mathcal{L}^{1,p}(D)$ . A function  $u \in \mathcal{L}_{loc}^{1,2}(D)$  is called *harmonic* in  $D$  if for any  $\phi \in C_0^\infty(D)$  one has

$$\int_D \sum_{j=1}^m X_j u X_j \phi \, dx = 0,$$

i.e., a harmonic function is a weak solution to the equation  $\mathcal{L}u = \sum_{j=1}^m X_j^* X_j u = 0$ . By Hörmander's hypoellipticity theorem [H], if  $u$  is harmonic in  $D$ , then  $u \in C^\infty(D)$ . Given a bounded open set  $D \subset \mathbb{R}^n$ , and a function  $\phi \in \mathcal{L}^{1,2}(D)$ , the Dirichlet problem consists in finding  $u \in \mathcal{L}_{loc}^{1,2}(D)$  such that

$$(3.1) \quad \begin{cases} \mathcal{L}u = 0 & \text{in } D, \\ u - \phi \in \mathcal{L}_0^{1,2}(D). \end{cases}$$

By adapting classical arguments, see for instance [GT], one can show that there exists a unique solution  $u \in \mathcal{L}^{1,2}(D)$  to (3.1). If we assume, in addition, that  $\phi \in C(D)$ , in general we cannot say that the function  $u$  takes up the boundary value  $\phi$  with continuity. A Wiener type criterion for sub-Laplacians was proved in [NS]. Subsequently, using the Wiener series in [NS], Citti obtained in [Ci] an estimate of the modulus of continuity at the boundary of the solution of (3.1). In [D] an integral Wiener type estimate at the boundary was established for a general class of quasilinear equations having  $p$ -growth in the sub-gradient. Since such estimate is particularly convenient for the applications, we next state it for the special case  $p = 2$  of linear equations.

**Theorem 3.1.** *Let  $\phi \in \mathcal{L}^{1,2}(D) \cap C(\overline{D})$ . Consider the solution  $u$  to (3.1). There exist  $C = C(X) > 0$ , and  $R_o = R_o(D, X) > 0$ , such that given  $x_o \in \partial D$ , and  $0 < r < R < R_o/3$ , one has*

$$\begin{aligned} \text{osc } \{u, D \cap B_d(x_o, r)\} &\leq \text{osc } \{\phi, \partial D \cap \overline{B}_d(x_o, 2R)\} \\ &+ \text{osc } (\phi, \partial D) \exp \left\{ -C \int_r^R \left[ \frac{\text{cap}_X(D^c \cap \overline{B}_d(x_o, t), B_d(x_o, 2t))}{\text{cap}_X(\overline{B}_d(x_o, t), B_d(x_o, 2t))} \right] \frac{dt}{t} \right\}. \end{aligned}$$

In Theorem 3.1, given a condenser  $(K, \Omega)$ , we have denoted by  $\text{cap}_X(K, \Omega)$  its Dirichlet capacity with respect to the subelliptic energy  $\mathcal{E}_X(u) = \int_\Omega |Xu|^2 dx$  associated with the system  $X = \{X_1, \dots, X_m\}$ . For the relevant properties of such capacity we refer the reader to [D], [CDG4]. A point  $x_o \in \partial D$  is called regular if, for any  $\phi \in \mathcal{L}^{1,2}(D) \cap C(\overline{D})$ , one has

$$(3.2) \quad \lim_{x \rightarrow x_o} u(x) = \phi(x_o).$$

If every  $x_o \in \partial D$  is regular, we say that  $D$  is regular. Similarly to the classical case, in the study of the Dirichlet problem an important notion is that of generalized, or Perron-Wiener-Brelot (PWB) solution to (3.1). For operators of Hörmander type the construction of a PWB solution was carried in the pioneering work of Bony [B], where the author also proved that sub-Laplacians satisfy an elliptic type strong maximum principle. We state next one of the main results in [B] in a form which is suitable for our purposes.

**Theorem 3.2.** *Let  $D \subset \mathbb{R}^n$  be a connected, bounded open set, and  $\phi \in C(\partial D)$ . There exists a unique harmonic function  $H_\phi^D$  which solves (1.3) in the sense of Perron-Wiener-Brelot. Moreover,  $H_\phi^D$  satisfies*

$$(3.3) \quad \sup_D |H_\phi^D| \leq \sup_{\partial D} |\phi|.$$

Theorem 3.2 allows to define the harmonic measure  $d\omega^x$  for  $D$  evaluated at  $x \in D$  as the unique probability measure on  $\partial D$  such that for every  $\phi \in C(\partial D)$

$$H_\phi^D(x) = \int_{\partial D} \phi(y) d\omega^x(y), \quad x \in D.$$

A uniform Harnack inequality was established, independently, by several authors, see [X], [CGL], [L]: If  $u$  is  $\mathcal{L}$ -harmonic in  $D \subset \mathbb{R}^n$  and non-negative then there exists  $C, a > 0$  such that for each ball  $B(x, ar) \subset D$  one has

$$(3.4) \quad \sup_{B(x,r)} u \leq C \inf_{B(x,r)} u.$$

Using such Harnack principle one sees that for any  $x, y \in D$ , the measures  $d\omega^x$  and  $d\omega^y$  are mutually absolutely continuous. For the basic properties of the harmonic measure we refer the reader to the paper [CG1]. Here, it is important to recall that, thanks to the results in [B], [CG1], the following result of Brelot type holds.

**Theorem 3.3.** *A function  $\phi$  is resolutive if and only if  $\phi \in L^1(\partial D, d\omega^x)$ , for one (and therefore for all)  $x \in D$ .*

The following definition is particularly important for its potential-theoretic implications. In the sequel, given a condenser  $(K, \Omega)$ , we denote by  $\text{cap}(K, \Omega)$  the sub-elliptic capacity of  $K$  with respect to  $\Omega$ , see [D].

**Definition 3.4.** *An open set  $D \subset \mathbb{R}^n$  is called thin at  $x_o \in \partial D$ , if*

$$(3.5) \quad \liminf_{r \rightarrow 0} \frac{\text{cap}_X(D^c \cap \overline{B}_d(x_o, r), B_d(x_o, 2r))}{\text{cap}_X(\overline{B}_d(x_o, r), B_d(x_o, 2r))} > 0.$$

**Theorem 3.5.** *If a bounded open set  $D \subset \mathbb{R}^n$  is thin at  $x_o \in \partial D$ , then  $x_o$  is regular for the Dirichlet problem.*

**Proof.** If  $D$  is thin at  $x_o \in \partial D$ , then

$$\int_0^R \left[ \frac{\text{cap}_X(D^c \cap \overline{B}_d(x_o, t), B_d(x_o, 2t))}{\text{cap}_X(\overline{B}_d(x_o, t), B_d(x_o, 2t))} \right] \frac{dt}{t} = \infty.$$

Thanks to Theorem 3.1, the divergence of the above integral implies for  $0 < r < R/3$

$$\text{osc} \{u, D \cap B_d(x_o, r)\} \leq \text{osc} \{\phi, \partial D \cap \overline{B}_d(x_o, 2R)\}.$$

Letting  $R \rightarrow 0$  we infer the regularity of  $x_o$ . □

A useful, and frequently used, sufficient condition for regularity is provided by the following definition.

**Definition 3.6.** An open set  $\Omega \subset \mathbb{R}^n$  is said to have positive density at  $x_o \in \partial\Omega$ , if one has

$$\liminf_{r \rightarrow 0} \frac{|\Omega \cap B_d(x_o, r)|}{|B_d(x_o, r)|} > 0.$$

**Proposition 3.7.** If  $D^c$  has positive density at  $x_o$ , then  $D$  is thin at  $x_o$ .

**Proof.** We recall the Poincaré inequality

$$\int_{\Omega} |\phi|^2 dx \leq C (\text{diam}(\Omega))^2 \int_{\Omega} |X\phi|^2 dx,$$

valid for any bounded open set  $\Omega \subset \mathbb{R}^n$ , and any  $\phi \in C_o^1(\Omega)$ , where  $\text{diam}(\Omega)$  represents the diameter of  $\Omega$  with respect to the distance  $d(x, y)$ , and  $C = C(\Omega, X) > 0$ . From the latter, we obtain

$$(3.6) \quad \frac{\text{cap}_X(D^c \cap \overline{B}_d(x_o, r), B_d(x_o, 2r))}{\text{cap}_X(\overline{B}_d(x_o, r), B_d(x_o, 2r))} \geq \frac{C}{r^2} \frac{|D^c \cap \overline{B}_d(x_o, r)|}{\text{cap}_X(\overline{B}_d(x_o, r), B_d(x_o, 2r))}.$$

Now the capacity estimates in [D], [CDG3] give

$$C r^{Q-2} \leq \text{cap}_X(\overline{B}_d(x_o, r), B_d(x_o, 2r)) \leq C^{-1} r^{Q-2},$$

for some constant  $C = C(\Omega, X) > 0$ . Using these estimates in (3.6) we find

$$\frac{\text{cap}_X(D^c \cap \overline{B}_d(x_o, r), B_d(x_o, 2r))}{\text{cap}_X(\overline{B}_d(x_o, r), B_d(x_o, 2r))} \geq C^* \frac{|D^c \cap \overline{B}_d(x_o, r)|}{|B_d(x_o, r)|},$$

where  $C^* = C^*(\Omega, X) > 0$ . The latter inequality proves that if  $D^c$  has positive density at  $x_o$ , then  $D$  is thin at the same point. □

A basic example of a class of regular domains for the Dirichlet problem is provided by the (Euclidean)  $C^{1,1}$  domains in a Carnot group of step  $r = 2$ . It was proved in [CG1] that such domains possess a scale invariant region of non-tangential approach at every boundary point, hence they satisfy the positive density condition in Proposition 3.7. Thus, in particular, every such domain is regular for the Dirichlet problem for any fixed sub-Laplacian on the group. Another important example is provided by the non-tangentially accessible domains (NTA domains, henceforth) studied in [CG1]. Such domains constitute a generalization of those introduced by Jerison and Kenig in the Euclidean setting [JK], see Section 8.

**Definition 3.8.** Let  $D \subset \mathbb{R}^n$  be a bounded open set. For  $0 < \alpha \leq 1$ , the class  $\Gamma_d^{0,\alpha}(D)$  is defined as the collection of all  $f \in C(D) \cap L^\infty(D)$ , such that

$$\sup_{x,y \in D, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty.$$

We endow  $\Gamma_d^{0,\alpha}(D)$  with the norm

$$\|f\|_{\Gamma_d^{0,\alpha}(D)} = \|f\|_{L^\infty(D)} + \sup_{x,y \in D, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

The meaning of the symbol  $\Gamma_{loc}^{0,\alpha}(D)$  is the obvious one, that is,  $f \in \Gamma_{loc}^{0,\alpha}(D)$  if, for every  $\omega \subset\subset D$ , one has  $f \in \Gamma_d^{0,\alpha}(\omega)$ . If  $F \subset \mathbb{R}^n$  denotes a bounded closed set, by  $f \in \Gamma_d^{0,\alpha}(F)$  we mean that  $f$  coincides on the set  $F$  with a function  $g \in \Gamma_d^{0,\alpha}(D)$ , where  $D$  is a bounded open set containing  $F$ . The Lipschitz class  $\Gamma_d^{0,1}(D)$  has a special interest, due to its connection with the Sobolev space  $\mathcal{L}^{1,\infty}(D)$ . In fact, we have the following theorem of Rademacher-Stepanov type, established in [GN1], which will be needed in the proof of Lemma 6.1.

**Theorem 3.9.** (i) *Given a bounded open set  $U \subset \mathbb{R}^n$ , there exist  $R_o = R_o(U, X) > 0$ , and  $C = C(U, X) > 0$ , such that if  $f \in \mathcal{L}^{1,\infty}(B_d(x_o, 3R))$ , with  $x_o \in U$  and  $0 < R < R_o$ , then  $f$  can be modified on a set of  $dx$ -measure zero in  $\bar{B}_d = B_d(x_o, R)$ , so as to satisfy for every  $x, y \in \bar{B}_d(x_o, R)$*

$$|f(x) - f(y)| \leq C d(x, y) \|f\|_{\mathcal{L}^{1,\infty}(B_d(x_o, 3R))}.$$

*If, furthermore,  $f \in C^\infty(B_d(x_o, 3R))$ , then in the right-hand side of the previous inequality one can replace the term  $\|f\|_{\mathcal{L}^{1,\infty}(B_d(x_o, 3R))}$  with  $\|Xf\|_{L^\infty(B_d(x_o, 3R))}$ .*

(ii) *Vice-versa, let  $D \subset \mathbb{R}^n$  be an open set such that  $\sup_{x,y \in D} d(x, y) < \infty$ . If  $f \in \Gamma_d^{0,1}(D)$ , then  $f \in \mathcal{L}^{1,\infty}(D)$ .*

We note explicitly that part (i) of Theorem 3.9 asserts that every function  $f \in \mathcal{L}^{1,\infty}(B_d(x_o, 3R))$  has a representative which is Lipschitz continuous in  $B_d(x_o, R)$  with respect to the metric  $d$ , i.e., continuing to denote with  $f$  such representative, one has  $f \in \Gamma^{0,1}(B_d(x_o, R))$ . Part (ii) was also obtained independently in [FSS]. The following result was established in [D].

**Theorem 3.10.** *Let  $D \subset \mathbb{R}^n$  be a bounded open set which is thin at every  $x_o \in \partial D$ . If  $\phi \in \Gamma^{0,\beta}(\bar{D})$ , for some  $\beta \in (0, 1)$ , then there exists  $\alpha \in (0, 1)$ , with  $\alpha = \alpha(D, X, \beta)$ , such that*

$$\sup_{x,y \in \bar{D}, x \neq y} \frac{|H_\phi^D(x) - H_\phi^D(y)|}{d(x,y)^\alpha} < \infty.$$

Given a bounded open set  $D \subset \mathbb{R}^n$ , consider the positive Green function  $G(x, y) = G(y, x)$  for  $\mathcal{L}$  and  $D$ , constructed in [B]. For every fixed  $x \in D$ , one can represent  $G(x, \cdot)$  as follows

$$(3.7) \quad G(x, \cdot) = \Gamma(x, \cdot) - h_x, \quad \text{where } h_x = H_{\Gamma(x, \cdot)}^D.$$

Since, by Hörmander's hypo-ellipticity theorem,  $\Gamma(x, \cdot) \in C^\infty(\mathbb{R}^n \setminus \{x\})$ , we conclude that, if  $D$  is thin at every  $x_o \in \partial D$ , then there exists  $\alpha \in (0, 1)$  such that, for every  $\epsilon > 0$ , one has

$$(3.8) \quad G(x, \cdot) \in \Gamma_d^{0,\alpha}(\bar{D} \setminus B(x_o, \epsilon)).$$

We close this section with recalling an important consequence of the results of Kohn and Nirenberg [KN1] (see Theorem 4), and of Derridj [De1], [De2], about smoothness in the Dirichlet problem at non-characteristic points. We recall the following definition.

**Definition 3.11.** *Given a  $C^1$  domain  $D \subset \mathbb{R}^n$ , a point  $x_o \in \partial D$  is called characteristic for the system  $X = \{X_1, \dots, X_m\}$  if for  $j = 1, \dots, m$  one has*

$$\langle X_j(x_o), \mathbf{N}(x_o) \rangle = 0,$$

*where  $\mathbf{N}(x_o)$  indicates a normal vector to  $\partial D$  at  $x_o$ . We indicate with  $\Sigma = \Sigma_{D,X}$  the collection of all characteristic points. The set  $\Sigma$  is a closed subset of  $\partial D$ .*

**Theorem 3.12.** *Let  $D \subset \mathbb{R}^n$  be a  $C^\infty$  domain which is regular for (1.3). Consider the harmonic function  $H_\phi^D$ , with  $\phi \in C^\infty(\partial D)$ . If  $x_o \in \partial D$  is a non-characteristic point for  $\mathcal{L}$ , then there exists an open neighborhood  $V$  of  $x_o$  such that  $H_\phi^D \in C^\infty(\overline{D} \cap V)$ .*

**Remark 3.13.** *We stress that, as we indicated in the introduction, the conclusion of Theorem 3.12 fails in general at characteristic points. In fact, it fails so completely that even if the domain  $D$  and the boundary datum  $\phi$  are real analytic, in general the solution of the Dirichlet problem  $H_\phi^D$  may be not better than Hölder continuous up to the boundary, see Theorem 3.10. An example of such negative phenomenon in the Heisenberg group  $\mathbb{H}^n$  was constructed by Jerison in [J1]. The next section is dedicated to it. For a related example concerning the heat equation see [KN2].*

#### 4. The example of D. Jerison

Consider the Heisenberg group (discussed in the introduction) with its left-invariant generators (1.4) of its Lie algebra. Recall that  $\mathbb{H}^n$  is equipped with the non-isotropic dilations

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t),$$

whose infinitesimal generator is given by the vector field

$$\mathcal{Z} = \sum_{i=1}^n \left( x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) + 2 \frac{\partial}{\partial t}.$$

We say that a function  $u : \mathbb{H}^n \rightarrow \mathbb{R}$  is *homogeneous of degree  $\alpha \in \mathbb{R}$*  if for every  $(z, t) \in \mathbb{H}^n$  and every  $\lambda > 0$  one has

$$u(\delta_\lambda(z, t)) = \lambda^\alpha u(z, t).$$

One easily checks that if  $u \in C^1(\mathbb{H}^n)$  then  $u$  is homogeneous of degree  $\alpha$  if and only if

$$\mathcal{Z}u = \alpha u.$$

We also consider the vector field

$$(4.1) \quad \Theta = \sum_{i=1}^n \left( x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right),$$

which is the infinitesimal generator of the one-parameter group of transformations  $R_\theta : \mathbb{H}^n \rightarrow \mathbb{H}^n$ ,  $\theta \in \mathbb{R}$ , given by

$$R_\theta(z, t) = (e^{i\theta} z, t), \quad z = x + iy \in \mathbb{C}^n.$$

Notice that when  $n = 1$ , then in the  $z$ -plane  $R_\theta$  is simply a counterclockwise rotation of angle  $\theta$ , and in such case in the standard polar coordinates  $(r, \theta)$  in  $\mathbb{C}$  we have

$$\Theta = \frac{\partial}{\partial \theta}.$$

In the sequel we will tacitly identify  $z = x + iy \simeq (x, y) \in \mathbb{R}^{2n}$ , and so  $|z| = \sqrt{|x|^2 + |y|^2}$ . We note explicitly that in the real coordinates  $(x, y, t)$  the real part of the Kohn-Spencer sub-Laplacian (1.5) on  $\mathbb{H}^n$  is given by

$$\mathcal{L}_o = \sum_{i=1}^{2n} X_i^2 = \Delta_z + \frac{|z|^2}{4} \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} \Theta.$$

It is easy to see that if  $u$  has *cylindrical symmetry*, i.e., if

$$u(z, t) = f(|z|, t),$$

then

$$\Theta u \equiv 0.$$

Consider the gauge in  $\mathbb{H}^n$

$$N = N(z, t) = (|z|^4 + 16t^2)^{1/4} .$$

The following formula follows from an explicit calculation

$$(4.2) \quad \psi \stackrel{def}{=} |\nabla^H N|^2 = \frac{|z|^2}{N^2} , \quad \Delta_H N = \frac{Q-1}{N} ,$$

where

$$Q = 2n + 2$$

is the so-called *homogeneous dimension* associated with the non-isotropic dilations  $\{\delta_\lambda\}_{\lambda>0}$ . As a consequence of (4.2), if  $u = f \circ N$  for some function  $f : [0, \infty) \rightarrow \mathbb{R}$ , then one has the beautiful formula

$$(4.3) \quad \mathcal{L}_o u = \psi \left[ f''(N) + \frac{Q-1}{N} f'(N) \right] .$$

Since  $f(t) = t^{2-Q}$  satisfies the ode in the right-hand side of (4.3) one can show that a fundamental solution of  $-\mathcal{L}_o$  with pole at the group identity  $e = (0, 0) \in \mathbb{H}^n$  is given by

$$(4.4) \quad \Gamma(z, t) = \frac{C_Q}{N(z, t)^{Q-2}} , \quad (z, t) \neq e ,$$

where  $C_Q > 0$  needs to be appropriately chosen.

The following example due to D. Jerison [J1] shows that, even when the domain and the boundary data are real analytic, in general the solution to the subelliptic Dirichlet problem (1.3) may not be any better than  $\Gamma^{0, \alpha}$  near a characteristic boundary point. Consider the domain

$$\Omega_M = \{(z, t) \in \mathbb{H}^n \mid t > M|z|^2\} , \quad M \in \mathbb{R} .$$

Since  $\Omega_M$  is scale invariant with respect to  $\{\delta_\lambda\}_{\lambda>0}$  we might think of  $\Omega_M$  as the analogue of a *convex cone* ( $M \geq 0$ ), or a *concave cone* ( $M < 0$ ). Introduce the variable

$$\tau = \tau(z, t) = \frac{4t}{N^2} , \quad (z, t) \neq e .$$

It is clear that  $\tau$  is homogeneous of degree zero and therefore

$$\mathcal{Z}\tau = 0 .$$

Moreover, with  $\Theta$  as in (4.1), one easily checks that

$$\Theta\tau = 0 .$$

It is important to observe the level sets  $\{\tau = \gamma\}$  are constituted by the  $t$ -axis when  $\gamma = 1$ , and by the paraboloids

$$t = \frac{\gamma}{4\sqrt{1-\gamma^2}}|z|^2 ,$$

if  $|\gamma| < 1$ . Furthermore, the function  $\tau$  takes the constant value

$$\tau = \frac{4M}{\sqrt{1+16M^2}} ,$$

on  $\partial\Omega_M$ . We now consider a function of the form

$$(4.5) \quad v = v(z, t) = N^\alpha u(\tau) ,$$

where the number  $\alpha > 0$  will be appropriately chosen later on. One has the following result whose verification we leave to the reader.

**Proposition 4.1.** *For any  $\alpha > 0$  one has*

$$\begin{aligned}\mathcal{L}_o v &= 4\psi N^{\alpha-2} \left\{ (1-\tau^2)u''(\tau) - \frac{Q}{2}\tau u'(\tau) + \frac{\alpha(\alpha+Q-2)}{4}u(\tau) \right\} \\ &= 4\psi N^{\alpha-2} \left\{ (1-\tau^2)u''(\tau) - (n+1)\tau u'(\tau) + \frac{\alpha(\alpha+2n)}{4}u(\tau) \right\}.\end{aligned}$$

Using Proposition 4.1 we can now construct a positive harmonic function in  $\Omega_M$  which vanishes on the boundary (this function is a Green function with pole at an interior point).

**Proposition 4.2.** *For any  $\alpha \in (0, 1]$  there exists a number  $M = M(\alpha) < 0$  such that the nonconvex cone  $\Omega_M$  admits a positive solution of  $\mathcal{L}_o v = 0$  of the form (4.5) which vanishes on  $\partial\Omega_M$ .*

**Proof.** From Proposition 4.1 we see that if  $v$  of the form (4.5) has to solve the equation  $\mathcal{L}_o v = 0$ , then the function  $u$  must be a solution of the Jacobi equation

$$(4.6) \quad (1-\tau^2)u''(\tau) - (n+1)\tau u'(\tau) + \frac{\alpha(\alpha+2n)}{4}u(\tau) = 0.$$

As we have observed the level  $\{\tau = 1\}$  is degenerate and corresponds to the  $t$ -axis  $\{z = 0\}$ . One solution of (4.6) which is smooth as  $\tau \rightarrow 1$  (remember, the  $t$ -axis is inside  $\Omega_M$  and thus we want our function  $v$  to be smooth around the  $t$ -axis since by hypoellipticity  $v$  has to be in  $C^\infty(\Omega_M)$ ) is the hypergeometric function

$$g_\alpha(\tau) = F\left(-\frac{\alpha}{2}, n + \frac{\alpha}{2}; \frac{n+1}{2}; \frac{1-\tau}{2}\right).$$

When  $0 < \alpha < 2$  one can verify that

$$g_\alpha(1) = 1, \quad \text{and that } g_\alpha(\tau) \rightarrow -\infty \text{ as } \tau \rightarrow -1^+.$$

Therefore,  $g_\alpha$  has a zero  $\tau_\alpha$ . One can check (see Erdelyi, Magnus, Oberhettinger and Tricomi, vol.1, p.110 (14)), that as  $\alpha \rightarrow 0^+$ , then  $\tau_\alpha \rightarrow -1^+$ . We infer that for  $\alpha > 0$  sufficiently close to 0 there exists  $-1 < \tau_\alpha < 0$  such that

$$g_\alpha(\tau_\alpha) = 0.$$

If we choose

$$M = M(\alpha) = \frac{\tau_\alpha}{\sqrt{1-\tau_\alpha^2}} < 0,$$

then it is clear that on  $\partial\Omega_M$  we have  $\tau \equiv \tau_\alpha$ , and therefore the function  $v$  of the form (6.10), with  $u(\tau) = g_\alpha(\tau)$ , has the property of being harmonic and nonnegative in  $\Omega$ , and furthermore on  $\partial\Omega_M$  we have that  $v = N^\alpha g_\alpha(\tau_\alpha) \equiv 0$ . This completes the proof.  $\square$

Since  $\alpha$  belongs the interval  $(0, 1)$ , then it is clear that  $v = N^\alpha(z, t)g_\alpha(\tau)$  belongs at most to the Folland-Stein Hölder class  $\Gamma^{0, \alpha}(\overline{\Omega}_M)$ , but is not any better than metrically Hölder in any neighborhood of  $e = (0, 0)$ . What produces this negative phenomenon is the fact that the point  $e \in \partial\Omega_M$  is characteristic for  $\Omega_M$ .

## 5. Subelliptic interior Schauder estimates

In this section we establish some basic interior Schauder type estimates that, besides from playing an important role in the sequel, also have an obvious independent interest. Such estimates are tailored on the intrinsic geometry of the system  $X = \{X_1, \dots, X_m\}$ , and are obtained by means of a family of sub-elliptic mollifiers which were introduced in [CDG1], see also [CDG2]. For convenience, we state the relevant results in terms of the  $X$ -balls  $B(x, r)$  introduced in Definition 2.2, but we stress that, thanks to (2.8), we could have as well employed the metric balls



$B_d(x, r)$ . Since in this paper our focus is on  $\mathcal{L}$ -harmonic functions, we do not explicitly treat the non-homogeneous equation  $\mathcal{L}u = f$  with a non-zero right-hand side. Estimates for solutions of the latter equation can, however, be obtained by relatively simple modifications of the arguments in the homogeneous case.

The following is the main result in this section.

**Theorem 5.1.** *Let  $D \subset \mathbb{R}^n$  be a bounded open set and suppose that  $u$  is harmonic in  $D$ . There exists  $R_o > 0$ , depending on  $D$  and  $X$ , such that for every  $x \in D$  and  $0 < r \leq R_o$  for which  $\overline{B}(x, r) \subset D$ , one has for any  $s \in \mathbb{N}$*

$$|X_{j_1} X_{j_2} \dots X_{j_s} u(x)| \leq \frac{C}{r^s} \max_{\overline{B}(x, r)} |u|,$$

for some constant  $C = C(D, X, s) > 0$ . In the above estimate, for every  $i = 1, \dots, s$ , the index  $j_i$  runs in the set  $\{1, \dots, m\}$ .

**Remark 5.2.** *We emphasize that Theorem 5.1 cannot be established similarly to its classical ancestor for harmonic functions, where one uses the mean-value theorem coupled with the trivial observation that any derivative of a harmonic function is harmonic. In the present non-commutative setting, derivatives of harmonic functions are no longer harmonic!*

A useful consequence of Theorem 5.1 is the following.

**Corollary 5.3.** *Let  $D \subset \mathbb{R}^n$  be a bounded, open set and suppose that  $u$  is a non-negative harmonic function in  $D$ . There exists  $R_o > 0$ , depending on  $D$  and  $X$ , such that for any  $x \in D$  and  $0 < r \leq R_o$  for which  $\overline{B}(x, 2r) \subset D$ , one has for any given  $s \in \mathbb{N}$*

$$|X_{j_1} X_{j_2} \dots X_{j_s} u(x)| \leq \frac{C}{r^s} u(x),$$

for some  $C = C(D, X, s) > 0$ .

**Proof.** Since  $u \geq 0$ , we immediately obtain the result from Theorem 5.1 and from the Harnack inequality (3.4).  $\square$

To prove Theorem 5.1, we use the family of sub-elliptic mollifiers introduced in [CDG1], see also [CDG2]. Choose a nonnegative function  $f \in C_o^\infty(\mathbb{R})$ , with  $\text{supp } f \subset [1, 2]$ , and such that  $\int_{\mathbb{R}} f(s) ds = 1$ , and let  $f_R(s) = R^{-1} f(R^{-1}s)$ . We define the kernel

$$K_R(x, y) = f_R\left(\frac{1}{\Gamma(x, y)}\right) \frac{|X_y \Gamma(x, y)|^2}{\Gamma(x, y)^2}.$$

Given a function  $u \in L_{loc}^1(\mathbb{R}^n)$ , following [CDG1] we define the *subelliptic mollifier* of  $u$  by

$$(5.1) \quad J_R u(x) = \int_{\mathbb{R}^n} u(y) K_R(x, y) dy, \quad R > 0.$$

We note that for any fixed  $x \in \mathbb{R}^n$ ,

$$(5.2) \quad \text{supp } K_R(x, \cdot) \subset \Omega(x, 2R) \setminus \Omega(x, R).$$

One of the important features of  $J_R u$  is expressed by the following theorem.

**Theorem 5.4.** *Let  $D \subset \mathbb{R}^n$  be open and suppose that  $u$  is harmonic in  $D$ . There exists  $R_o > 0$ , depending on  $D$  and  $X$ , such that for any  $x \in D$ , and every  $0 < R \leq R_o$  for which  $\overline{\Omega}(x, 2R) \subset D$ , one has*

$$u(x) = J_R u(x).$$

**Proof.** Let  $u$  and  $\Omega(x, R)$  be as in the statement of the theorem. We obtain for  $\psi \in C^\infty(D)$  and  $0 < t \leq R$ , see [CGL],

$$(5.3) \quad \psi(x) = \int_{\partial\Omega(x,t)} \psi(y) \frac{|X_y \Gamma(x, y)|^2}{|D\Gamma(x, y)|} dH_{n-1}(y) + \int_{\Omega(x,t)} \mathcal{L}\psi(y) \left[ \Gamma(x, y) - \frac{1}{t} \right] dy.$$

Taking  $\psi = u$  in (5.3), we find

$$(5.4) \quad u(x) = \int_{\partial\Omega(x,t)} u(y) \frac{|X_y \Gamma(x, y)|^2}{|D\Gamma(x, y)|} dH_{n-1}(y).$$

We are now going to use (5.4) to complete the proof. The idea is to start from the definition of  $J_R u(x)$ , and then use Federer co-area formula [Fe]. One finds

$$J_R u(x) = \int_0^\infty f_R(t) \left[ \int_{\partial\Omega(x,t)} u(y) \frac{|X_y \Gamma(x, y)|^2}{|D\Gamma(x, y)|} dH_{n-1}(y) \right] dt.$$

The previous equality, (5.4), and the fact that  $\int_{\mathbb{R}} f_R(s) ds = 1$ , imply the conclusion.  $\square$

The essence of our main a priori estimate is contained in the following theorem.

**Theorem 5.5.** *Fix a bounded set  $U \subset \mathbb{R}^n$ . There exists a constant  $R_o > 0$ , depending only on  $U$  and on the system  $X$ , such that for any  $u \in L^1_{loc}(\mathbb{R}^n)$ ,  $x \in U$ ,  $0 < R \leq R_o$ , and  $s \in \mathbb{N}$  one has for some  $C = C(U, X, s) > 0$ ,*

$$|X_{j_1} X_{j_2} \dots X_{j_s} J_R u(x)| \leq \frac{C}{R} \frac{1}{F(x, R)^{2+s}} \int_{\Omega(x, R)} |u(y)| dy.$$

**Proof.** We first consider the case  $s = 1$ . From (2.7), and from the support property (5.2) of  $K_R(x, \cdot)$ , we can differentiate under the integral sign in (5.1), to obtain

$$|X J_R u(x)| \leq \int_{B(x, 2R)} |u(y)| |X_x K_R(x, y)| dy.$$

By the definition of  $K_R(x, y)$  it is easy to recognize that the components of its sub-gradient  $X_x K_R(x, y)$  are estimated as follows

$$\begin{aligned} |X_j K_R(x, y)| &\leq C R^{-2} |X\Gamma(x, y)|^3 \Gamma(x, y)^{-4} \\ &+ C R^{-1} \Gamma(x, y)^{-2} \sum_{k=1}^m |X_j X_k \Gamma(x, y)| |X_k \Gamma(x, y)| \\ &+ C R^{-1} |X\Gamma(x, y)|^3 \Gamma(x, y)^{-3} \\ &= I_R^1(x, y) + I_R^2(x, y) + I_R^3(x, y). \end{aligned}$$

To control the three terms in the right-hand side of the above inequality, we use the size estimates (2.7), along with the observation that, due to the fact that on the support of  $K_R(x, \cdot)$  one has

$$\frac{1}{2R} < \Gamma(x, y) \leq \frac{1}{R},$$

then Theorem 2.3, and (2.9), give for all  $x \in U$ ,  $0 < R \leq R_o$ , and  $y \in \Omega(x, 2R) \setminus \Omega(x, R)$

$$(5.5) \quad C \leq \frac{d(x, y)}{F(x, R)} \leq C^{-1}.$$

Using (2.7), (5.5), one obtains that for  $i = 1, 2, 3$

$$\sup_{y \in \Omega(x, 2R) \setminus \Omega(x, R)} |I_R^i(x, y)| \leq \frac{C}{R F(x, R)^3}$$

for any  $x \in U$ , provided that  $0 < R \leq R_o$ . This completes the proof in the case  $s = 1$ . The case  $s \geq 2$  is handled recursively by similar considerations based on Theorem 2.3, and we omit details. It may be helpful for the interested reader to note that Theorem 2.3 implies

$$|X_{j_1} X_{j_2} \dots X_{j_s} \Gamma(x, y)| \leq C d(x, y)^{-s} \Gamma(x, y),$$

so that by (5.5) one obtains

$$(5.6) \quad \sup_{y \in \Omega(x, 2R) \setminus \Omega(x, R)} |X_{j_1} X_{j_2} \dots X_{j_s} \Gamma(x, y)| \leq \frac{C}{RF(x, R)^s}.$$

□

We are finally in a position to prove Theorem 5.1.

**Proof of Theorem 5.1.** We observe explicitly that the assumption states that with  $R = E(x, r)/2$ , then  $\overline{\Omega}(x, 2R) = \overline{B}(x, r) \subset D$ . By Theorem 5.4, and by (5.6), we find

$$\begin{aligned} |X_{j_1} X_{j_2} \dots X_{j_s} u(x)| &= |X_{j_1} X_{j_2} \dots X_{j_s} (J_R u)(x)| \\ &\leq \frac{C}{RF(x, R)^{2+s}} \int_{\Omega(x, R)} |u(y)| dy \leq C \frac{|\Omega(x, R)|}{RF(x, R)^{2+s}} \max_{\overline{\Omega}(x, R)} |u|. \end{aligned}$$

To complete the proof we only need to observe that  $\Omega(x, R) = B(x, r)$ , and that, thanks to Theorem 2.3, (2.9), one has

$$\frac{C}{r^s} \leq \frac{|B(x, r)|}{RF(x, R)^{2+s}} \leq \frac{C^{-1}}{r^s}.$$

□

**Remark 5.6.** We observe explicitly that when  $\mathbb{G}$  is a Carnot group with  $X_1, \dots, X_m$  being a fixed basis of the horizontal layer of its Lie algebra, then the constant  $C$  in Theorem 5.1 and Corollary 5.3 can be taken independent of the open set  $D$ .

## 6. Lipschitz boundary estimates for the Green function

In this section we establish some basic estimates at the boundary for the Green function associated to a sub-Laplacian, when the relevant domain possesses an appropriate analogue of the outer tangent sphere condition introduced by Poincaré in his famous paper [P]. Analyzing the domain  $\Omega_M$  in Remark 3.13 one recognizes that Jerison's negative example fails to possess a tangent outer gauge sphere at its characteristic point. We thus conjectured that by imposing such condition one should be able to establish the boundedness near the boundary of the horizontal gradient of the Green function (see for instance [G] for the classical case of elliptic or parabolic operators). This intuition has proved correct. In their paper [LU1] Lanconelli and Uguzzoni have proved the boundedness of the Poisson kernel for a domain satisfying the outer sphere condition in the Heisenberg group, whereas in [CGN2] a similar result was successfully combined with those in [CG1] to obtain a complete solution of the Dirichlet problem for a large class of domains in groups of Heisenberg type.

The objective of this section is to generalize the cited results in [LU1] and [CGN2] to the Poisson kernel associated with an operator of Hörmander type. Namely, *if  $D \subset \mathbb{R}^n$  is a bounded domain satisfying an intrinsic uniform outer sphere condition with respect to a system  $X = \{X_1, \dots, X_m\}$  satisfying (1.1), and having Green function  $G(x, y) = G_D(x, y)$ , if we fix the singularity at an interior point  $x_1 \in D$ , then the function  $x \rightarrow |XG(x_1, x)|$ , which is well defined for  $x \in D \setminus \{x_1\}$ , belongs to  $L^\infty$  in a neighborhood of  $\partial D$ .* The exact statements are contained in Corollaries 6.7 and 6.11.

We emphasize that, in view of Theorem 3.12, the main novelty of this result lies in that we do allow the boundary point to be characteristic. As it will be clear from the analysis below,

the passage from the group setting to the case of general sub-Laplacians involves overcoming various non-trivial obstacles.

Our first task is to obtain a growth estimate at the boundary for harmonic functions which vanish on a distinguished portion of the latter. We show that any such function grows at most linearly with respect to the Carnot-Carathéodory distance associated to the system  $X$ . The proof of this result ultimately relies on delicate estimates of a suitable barrier whose construction is inspired to that given by Poincaré [P], see also [G]. We begin with a lemma which plays a crucial role in the sequel. The function  $\Gamma(x, y) = \Gamma(y, x)$  denotes the positive fundamental solution of the sub-Laplacian associated with the system  $X$ , see Section 2.

**Lemma 6.1.** *For any bounded set  $U \subset \mathbb{R}^n$ , there exist  $R_o, C > 0$ , depending on  $U$  and  $X$ , such that for every  $x_o \in U$ , and  $x, y \in \mathbb{R}^n \setminus B_d(x_o, r)$ , one has*

$$|\Gamma(x_o, x) - \Gamma(x_o, y)| \leq C \frac{r}{|B_d(x_o, r)|} d(x, y).$$

**Proof.** We distinguish two cases: (i)  $d(x, y) > \theta r$ ; (ii)  $d(x, y) \leq \theta r$ . Here,  $\theta \in (0, 1)$  is to be suitably chosen. Case (i) is easy. Using (2.7) we find

$$\begin{aligned} |\Gamma(x_o, x) - \Gamma(x_o, y)| &\leq \Gamma(x_o, x) + \Gamma(x_o, y) \\ &\leq C \left[ \frac{d(x_o, x)^2}{|B_d(x_o, d(x_o, x))|} + \frac{d(x_o, y)^2}{|B_d(x_o, d(x_o, y))|} \right] \\ &\leq C \left\{ \frac{1}{E(x_o, d(x_o, x))} + \frac{1}{E(x_o, d(x_o, y))} \right\} \leq C \frac{1}{E(x_o, r)} < \frac{C}{\theta} \frac{r}{|B_d(x_o, r)|} d(x, y). \end{aligned}$$

We next consider case (ii), and let  $\rho = d(x, y) \leq \theta r$ . Let  $\gamma$  be a sub-unitary curve joining  $x$  to  $y$  with length  $l_s(\gamma) \leq \rho + \rho/16$ . The existence of such a curve is guaranteed by the definition of  $d(x, y)$ . Consider the function  $g(P) \stackrel{\text{def}}{=} d(x, P) - d(y, P)$ . By the continuity of  $g : \{\gamma\} \rightarrow \mathbb{R}$ , and by the intermediate value theorem, we can find  $P \in \{\gamma\}$  such that  $d(x, P) = d(y, P)$ . For such point  $P$ , we must have

$$(6.1) \quad d(x, P) = d(y, P) \leq \frac{3}{4} \rho.$$

If (6.1) were not true, we would in fact have

$$\frac{3}{4}\rho + \frac{3}{4}\rho < d(x, P) + d(y, P) \leq l_s(\gamma) \leq \rho + \frac{\rho}{16},$$

which is a contradiction. From (6.1) we conclude that  $x, y \in B_d(P, 3\rho/4)$ . Moreover,

$$d(P, x_o) \geq d(x, x_o) - d(x, P) \geq r - \frac{3}{4}\rho \geq \left(1 - \frac{3}{4}\theta\right) r.$$

We claim that

$$(6.2) \quad B_d(P, \frac{9}{4}\rho) \subset \mathbb{R}^n \setminus B_d(x_o, r/2),$$

provided that we take  $\theta = \frac{1}{6}$ . In fact, let  $z \in B_d(P, \frac{9}{4}\rho)$ , then

$$d(z, x_o) \geq d(P, x_o) - d(z, P) \geq \left(1 - \frac{3}{4}\theta\right) r - \frac{9}{4}\theta r = \left(1 - \frac{3}{4}\theta - \frac{9}{4}\theta\right) r = \frac{r}{2}.$$

This proves (6.2). The above considerations allow to apply Theorem 3.9, which, keeping in mind that  $\Gamma(x_o, \cdot) \in C^\infty(B_d(P, \frac{9}{4}\rho))$ , presently gives

$$(6.3) \quad |\Gamma(x_o, x) - \Gamma(x_o, y)| \leq C \rho \sup_{\xi \in B_d(P, \frac{9}{4}\rho)} |X\Gamma(x_o, \xi)|.$$

Using (2.7) we obtain for  $\xi \in B_d(P, \frac{9}{4}\rho)$

$$|X\Gamma(x_o, \xi)| \leq C \frac{1}{d(x_o, \xi) E(x_o, d(x_o, \xi))},$$

where  $t \rightarrow E(x_o, t)$  is the function introduced in (2.6). Since by (6.2) we have  $d(x_o, \xi) \geq r/2$ , the latter estimate, combined with the increasingness of  $E(x_o, \cdot)$ , leads to the conclusion

$$\sup_{\xi \in B_d(P, \frac{9}{4}\rho)} |X\Gamma(x_o, \xi)| \leq C \frac{1}{rE(x_o, r)}.$$

Inserting this inequality in (6.3), and observing that  $\frac{1}{rE(x_o, r)} \leq C \frac{r}{|B_d(x_o, r)|}$ , we find

$$|\Gamma(x_o, x) - \Gamma(x_o, y)| \leq C \frac{r}{|B_d(x_o, r)|} d(x, y).$$

This completes the proof of the lemma.  $\square$

The following definition plays a crucial role in the subsequent development.

**Definition 6.2.** A domain  $D \subset \mathbb{R}^n$  is said to possess an outer  $X$ -ball tangent at  $x_o \in \partial D$  if for some  $r > 0$  there exists a  $X$ -ball  $B(x_1, r)$  such that:

$$(6.4) \quad x_o \in \partial B(x_1, r), \quad B(x_1, r) \cap D = \emptyset.$$

We say that  $D$  possesses the uniform outer  $X$ -ball if one can find  $R_o > 0$  such that for every  $x_o \in \partial D$ , and any  $0 < r < R_o$ , there exists a  $X$ -ball  $B(x_1, r)$  for which (6.4) holds.

Some comments are in order. First, it should be clear from (2.8) that the existence of an outer  $X$ -ball tangent at  $x_o \in \partial D$  implies that  $D$  is thin at  $x_o$  (the reverse implication is not necessarily true). Therefore, thanks to Theorem 3.5,  $x_o$  is regular for the Dirichlet problem. Secondly, when  $X = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ , then the distance  $d(x, y)$  is just the ordinary Euclidean distance  $|x - y|$ . In such case, Definition 6.2 coincides with the notion introduced by Poincaré in his classical paper [P]. In this setting a  $X$ -ball is just a standard Euclidean ball, then every  $C^{1,1}$  domain and every convex domain possess the uniform outer  $X$ -ball condition. When we abandon the Euclidean setting, the construction of examples is technically much more involved and we discuss them in the last section of this paper.

We are now ready to state the first key boundary estimate.

**Theorem 6.3.** Let  $D \subset \mathbb{R}^n$  be a connected open set, and suppose that for some  $r > 0$ ,  $D$  has an outer  $X$ -ball  $B(x_1, r)$  tangent at  $x_o \in \partial D$ . There exists  $C > 0$ , depending only on  $D$  and on  $X$ , such that if  $\phi \in C(\partial D)$ ,  $\phi \equiv 0$  in  $B(x_1, 2r) \cap \partial D$ , then we have for every  $x \in D$

$$|H_\phi^D(x)| \leq C \frac{d(x, x_o)}{r} \max_{\partial D} |\phi|.$$

**Proof.** Without loss of generality we assume  $\max_{\partial D} |\phi| = 1$ . Following the idea in [P] we introduce the function

$$(6.5) \quad f(x) = \frac{E(x_1, r)^{-1} - \Gamma(x_1, x)}{E(x_1, r)^{-1} - E(x_1, 2r)^{-1}}, \quad x \in D,$$

where  $x \rightarrow \Gamma(x_1, x)$  denotes the positive fundamental solution of  $\mathcal{L}$ , with singularity at  $x_1$ , and  $t \rightarrow E(x_1, t)$  is defined as in (2.6). Clearly,  $f$  is  $\mathcal{L}$ -harmonic in  $\mathbb{R}^n \setminus \{x_1\}$ . Since  $\Gamma(x_1, \cdot) \leq E(x_1, r)^{-1}$  outside  $B(x_1, r)$ , we see that  $f \geq 0$  in  $\mathbb{R}^n \setminus B(x_1, r)$ , hence in particular in  $D$ . Moreover,  $f \equiv 1$  on  $\partial B(x_1, 2r) \cap D$ , whereas  $f \geq 1$  in  $(\mathbb{R}^n \setminus B(x_1, 2r)) \cap \overline{D}$ . By Theorem 3.2 we infer

$$|H_\phi^D(x)| \leq f(x) \quad \text{for every } x \in D.$$

The proof will be completed if we show that

$$(6.6) \quad f(x) \leq C \frac{d(x, x_o)}{r}, \quad \text{for every } x \in D.$$

Consider the function  $h(t) = E(x_1, t)^{-1}$ . We have for  $0 < s < t < R_o$ ,

$$h(s) - h(t) = (t - s) \frac{E'(x_1, \tau)}{E(x_1, \tau)^2},$$

for some  $s < \tau < t$ . Using the increasingness of the function  $r \rightarrow rE(x_1, r)$ , which follows from that of  $E(x_1, \cdot)$ , and the crucial estimate

$$C \leq \frac{rE'(x_1, r)}{E(x_1, r)} \leq C^{-1},$$

which is readily obtained from the definition of  $\Lambda(x_1, r)$  in (2.2), we find

$$(6.7) \quad C \frac{t - s}{tE(x_1, t)} \leq h(s) - h(t) \leq C^{-1} \frac{t - s}{sE(x_1, s)}.$$

Keeping in mind the definition (6.5) of  $f$ , from (6.7), and from the fact that  $E(x_1, \cdot)$  is doubling, we obtain

$$f(x) \leq C E(x_1, r) \{ \Gamma(x_1, x_o) - \Gamma(x_1, x) \},$$

where we have used the hypothesis that  $x_o \in \partial B(x_1, r)$ . The proof of (6.6) will be achieved if we show that for  $x \in \mathbb{R}^n \setminus B(x_1, r)$

$$\Gamma(x_1, x_o) - \Gamma(x_1, x) \leq C d(x, x_o) \frac{1}{rE(x_1, r)}.$$

In view of (2.8), the latter inequality follows immediately from Lemma 6.1. This completes the proof.  $\square$

Let  $D \subset \mathbb{R}^n$  be a domain. Consider the positive Green function  $G(x, y)$  associated to  $\mathcal{L}$  and  $D$ . From Theorem 3.2 and from the estimates (2.7) one easily sees that there exists a positive constant  $C_D$  such that for every  $x, y \in D$

$$(6.8) \quad 0 \leq G(x, y) \leq C_D \frac{d(x, y)^2}{|B_d(x, d(x, y))|},$$

for each  $x, y \in D$ . Our next task is to obtain more refined estimates for  $G$ .

**Theorem 6.4.** *Suppose that  $D \subset \mathbb{R}^n$  satisfy the uniform outer  $X$ -ball condition. There exists a constant  $C = C(X, D) > 0$  such that*

$$G(x, y) \leq C \frac{d(x, y)}{|B_d(x, d(x, y))|} d(y, \partial D)$$

for each  $x, y \in D$ , with  $x \neq y$ .

**Proof.** Consider  $a > 1$  as in (2.8), and let  $R_o$  be the constant in Definition 6.2 of uniform outer  $X$ -ball condition. The estimate that we want to prove is immediate if one of the points is away from the boundary. In fact, if either  $d(y, \partial D) \geq \frac{d(x, y)}{a^2(3+a)}$ , or  $d(y, \partial D) \geq R_o$ , then the conclusion follows from (6.8). We may thus assume that

$$(6.9) \quad a d(y, \partial D) < \frac{d(x, y)}{a(a+3)}, \quad \text{and} \quad d(y, \partial D) < R_o.$$

We now choose

$$r = \min \left( \frac{d(x, y)}{2a(a+3)}, \frac{aR_o}{2} \right).$$

One easily verifies from (6.9) that  $ad(y, \partial D) < 2r$ . Let  $x_o$  be the point in  $\partial D$  such that  $d(y, \partial D) = d(y, x_o)$  and consider the outer  $X$ -ball  $B(x_1, r/a)$  tangent to the boundary of  $D$  in  $x_o$ . We claim that

$$y \in D \cap B(x_1, (a+3)r).$$

To see this observe that by (2.8)  $x_o \in \overline{B}(x_1, \frac{r}{a}) \subset \overline{B}_d(x_1, r)$ , and therefore

$$d(y, x_1) \leq d(y, x_o) + d(x_o, x_1) = d(y, \partial D) + d(x_o, x_1) \leq \frac{a+2}{a}r < \frac{a+3}{a}r.$$

This shows  $y \in B_d(x_1, a^{-1}(a+3)r)$ . Another application of (2.8) implies the claim. Next, the triangle inequality gives

$$d(x, x_1) \geq d(x, y) - d(x_1, y) \geq d(x, y) - \frac{a+3}{a}r \geq d(x, y)(1 - \frac{1}{2a^2}),$$

and consequently

$$x \in \mathbb{R}^n \setminus B_d(x_1, (1 - \frac{1}{2a^2})d(x, y)).$$

On the other hand (2.8) implies

$$\mathbb{R}^n \setminus B_d(x_1, (1 - \frac{1}{2a^2})d(x, y)) \subset \mathbb{R}^n \setminus B(x_1, \frac{1}{a}(1 - \frac{1}{2a^2})d(x, y)) \subset \mathbb{R}^n \setminus B(x_1, (a+3)r),$$

the last inclusion being true since  $a > 1$ .

We now consider the Perron-Wiener-Brelot solution  $v$  to the Dirichlet problem  $\mathcal{L}v = 0$  in  $B(x_1, (a+3)r) \cap D$ , with boundary datum a function  $\phi \in C(\partial(B(x_1, (a+3)r) \cap D))$ , such that  $0 \leq \phi \leq 1$ ,  $\phi = 1$  on  $\partial B(x_1, (3+a)r) \cap D$ , and  $\phi = 0$  on  $\partial D \cap B(x_1, (1+a)r)$ . We observe in passing that, thanks to the assumptions on  $D$ , we can only say that  $v$  is continuous up to the boundary in that portion of  $\partial(B(x_1, (a+3)r) \cap D)$  that is common to  $\partial D$ . However such continuity is not needed to implement Theorem 3.2 and deduce that  $0 \leq v \leq 1$ . We observe that  $D \cap B(x_1, (a+3)r)$  satisfies the outer  $\mathcal{L}$ -ball condition at the point  $x_o \in \partial D$ . Applying Theorem 6.3 one infers for every  $y \in D \cap B(x_1, (a+3)r)$

$$(6.10) \quad |v(y)| \leq C \frac{d(y, \partial D)}{r}.$$

Let  $C_D$  be as in (6.8) and define  $w(z) = C_D^{-1}E(x, \beta d(x, y))G(x, z)$ , where  $\beta = (1 - \frac{1}{2a^2} - \frac{1}{2a})$ . Since  $x \notin B(x_1, (a+3)r)$ , then  $\mathcal{L}w = 0$  in  $B(x_1, (a+3)r) \cap D$ . Observe that if  $z \in \partial B(x_1, (a+3)r)$ , then

$$d(x, z) \geq d(x, x_1) - d(z, x_1) \geq (1 - \frac{1}{2a^2}) - (a+3)r \geq \beta d(x, y),$$

from our choice of  $r$  and  $\beta$ . Consequently, in view of the monotonicity of  $r \rightarrow E(x, r)$  and (6.8), we have that  $w \leq C_D^{-1}E(x, d(x, z))G(x, z) \leq 1$  on  $\partial(B(x_1, (a+3)r) \cap D)$ . By Theorem 3.2 one concludes that  $w(y) \leq v(y)$  in  $D \cap B(x_1, (a+3)r)$ . The estimate of  $v$  established above, along with (2.1), completes the proof.  $\square$

It was observed in [LU2, Theorem 50] that in a Carnot group, by exploiting the symmetry of the Green function  $G(y, x) = G(x, y)$ , one can actually improve the estimate in Theorem 6.4 as follows

$$G(x, y) \leq C d(x, y)^{-Q} d(x, \partial D) d(y, \partial D), \quad x, y \in D, \quad x \neq y,$$

where  $Q$  represents the homogeneous dimension of the group. An analogous improvement can be obtained in the more general setting of this paper. To see this, note that the symmetry of  $G$  and the estimate in Theorem 6.4 give for every  $x, y \in D$

$$(6.11) \quad G(y, x) = G(x, y) \leq C \frac{d(x, y)}{|B_d(x, d(x, y))|} d(y, \partial D),$$

where  $C > 0$  is the constant in the statement of Theorem 6.4. We now argue exactly as in the case in which (6.9) holds in the proof of Theorem 6.4, except that we now define

$$w(z) = C^{-1} d(x, \partial D)^{-1} \frac{|B_d(x, d(x, y))|}{d(x, y)} G(z, x), \quad z \in \overline{B(x_1, (a+3)r) \cap D}.$$

Using (6.11) instead of (6.8) we reach the conclusion that

$$w(z) \leq 1, \quad \text{for every } z \in \partial(B(x_1, (a+3)r) \cap D).$$

Since  $\mathcal{L}w = 0$  in  $B(x_1, (a+3)r) \cap D$ , by Theorem 3.2 we conclude as before that  $w(y) \leq v(y)$  in  $D \cap B(x_1, (a+3)r)$ . Combining this estimate with (6.10) we have proved the following result.

**Corollary 6.5.** *Suppose that  $D \subset \mathbb{R}^n$  satisfy the uniform outer  $X$ -ball condition. There exists a constant  $C = C(X, D) > 0$  such that*

$$G(x, y) \leq C \frac{d(x, \partial D) d(y, \partial D)}{|B_d(x, d(x, y))|},$$

for each  $x, y \in D$ , with  $x \neq y$ .

We now turn to estimating the horizontal gradient of the Green function up to the boundary. The next result plays a central role in the rest of the paper.

**Theorem 6.6.** *Assume the uniform outer  $X$ -ball condition for  $D \subset \mathbb{R}^n$ . There exists a constant  $C = C(X, D) > 0$  such that*

$$|XG(x, y)| \leq C \frac{d(x, y)}{|B_d(x, d(x, y))|},$$

for each  $x, y \in D$ , with  $x \neq y$ .

**Proof.** Let  $R_o$  be as in Definition 6.2. Fix  $x, y \in D$  and choose  $0 < r < R_o$  such that  $x \notin B_d(y, ar) \subset \overline{D}$ . Applying Corollary 5.3 and (2.8) to  $G(x, \cdot)$  we obtain for every  $z \in B(y, r)$

$$|XG(x, z)| \leq \frac{C}{r} G(x, z).$$

If  $d(y, \partial D) \leq 2ad(x, y)$ , we choose  $r = \min\left(\frac{d(y, \partial D)}{2a}, \frac{R_o}{2}\right)$  and then the latter inequality implies the conclusion via Theorem 6.4. If  $d(y, \partial D) > 2ad(x, y)$ , then keeping in mind that  $G(x, \cdot) = \Gamma(x, \cdot) - h_x$ , we use (2.7) to bound  $|X\Gamma|$ , and, with  $r = \min\left(\frac{d(y, \partial D)}{2a}, \frac{R_o}{2}\right)$ , we apply Corollary 5.3 and the maximum principle to obtain

$$|Xh_x(y)| \leq \frac{C}{r} h_x(y) = \frac{C}{r} h_y(x) \leq \frac{C}{r} \sup_{w \in \partial D} \Gamma(y, w) = \frac{C}{r} \Gamma(y, z)$$

for some  $z \in \partial D$ . On the other hand, one has

$$d(x, y) < \frac{d(y, \partial D)}{2a} \leq \frac{d(y, z)}{2a}$$

so that using (2.7) one more time

$$\Gamma(y, z) \leq C \frac{1}{E(y, d(y, z))} \leq C \frac{1}{E(y, 2ad(x, y))} \leq C \Gamma(y, x) \leq C \frac{d(x, y)^2}{|B_d(x, d(x, y))|}.$$

Replacing this inequality in the estimate for  $|Xh_x(y)|$  we reach the desired conclusion.  $\square$

**Corollary 6.7.** *If  $D \subset \mathbb{R}^n$  satisfies the uniform outer  $X$ -ball condition, then for any  $x_o \in D$  and every open neighborhood  $U$  of  $\partial D$ , such that  $x_o \notin \overline{U}$ , one has  $G(x_o, \cdot) \in \mathcal{L}^{1, \infty}(U)$ . Moreover, its  $\mathcal{L}^{1, \infty}(U)$  norm depends on  $D, X$  and  $U$  but it is independent of  $x_o$ .*



**Localizing the hypothesis.** It is interesting to note that one can still prove that  $G(x_o, \cdot) \in \mathcal{L}^{1,\infty}(U)$  under the weaker hypothesis that the uniform outer  $X$ -ball condition be satisfied only in a neighborhood of the characteristic set of  $D$ . In this case, however, the uniform estimates in  $x_o$  will be lost. We devote the last part of this section to the proof of this result. Let  $\Sigma = \Sigma_D \subset \partial D$  denote the compact set of all characteristic points.

**Definition 6.8.** Let  $D$  be a  $C^1$  domain. We say that  $D$  possesses the uniform outer  $X$ -ball in a neighborhood of  $\Sigma$  if for a given choice of an open set  $V$  containing  $\Sigma$ , one can find  $R_o > 0$  such that for every  $Q \in V \cap \partial D$  and  $0 < r < R_o$  there exists a  $X$ -ball  $B(x_1, r)$  for which (6.4) holds. More in general, we say that  $D$  possesses the uniform outer  $X$ -ball along the set  $V \cap \partial D$  if one can find  $R_o > 0$  such that for every  $x_o \in V \cap \partial D$  and  $0 < r < R_o$  there exists a  $X$ -ball  $B(x_1, r)$  for which (6.4) holds.

Our first step consists in proving "localized" versions of Theorems 6.4 and 6.6.

**Theorem 6.9.** Let  $D \subset \mathbb{R}^n$  be a domain that is regular for the Dirichlet problem. Let  $P \in \partial D$  and assume that for some  $\epsilon > 0$  the set  $D$  possesses the uniform outer  $X$ -ball along  $B_d(P, 2\epsilon) \cap \partial D$ . There exists a constant  $C = C(X, D) > 0$  such that

$$G(x, y) \leq C \frac{d(x, y)}{|B_d(x, d(x, y))|} d(y, \partial D)$$

for each  $y \in B_d(P, \epsilon) \cap D$ , and  $x \in D$ , with  $x \neq y$ .

**Proof.** The proof follows closely the one of Theorem 6.4 and we will adopt the same notation as in that proof. Let  $x_o$  be the point in  $\partial D$  closest to  $y$ . In order to apply the arguments in the proof of Theorem 6.4 we need to show that the set  $D$  has an outer  $\mathcal{L}$ -ball  $B(x_1, r/a)$  at  $x_o$  for every  $0 < r < R_o$ . Given our hypothesis it suffices to show that  $x_o \in B_d(P, 2\epsilon) \cap \partial D$ . Observe that  $d(y, x_o) \leq d(y, P) < \epsilon$ , and consequently  $d(P, x_o) < 2\epsilon$ . Since  $D$  has an outer  $X$ -ball  $B(x_1, r/a)$  at  $x_o$  for every  $0 < r < R_o$ , then so does the subset  $B(x_1, (a+3)r) \cap D$ . The rest of the proof is a word by word repetition of the one for Theorem 6.4.  $\square$

**Theorem 6.10.** Let  $D \subset \mathbb{R}^n$  be a domain that is regular for the Dirichlet problem. Let  $P \in \partial D$  and assume that for some  $\epsilon > 0$  the set  $D$  possesses the uniform outer  $X$ -ball along  $B_d(P, 2\epsilon) \cap \partial D$ . There exists a constant  $C = C(X, D) > 0$  such that

$$|XG(x, y)| \leq C \frac{d(x, y)}{|B_d(x, d(x, y))|},$$

for each  $y \in B_d(P, \frac{1}{2}\epsilon) \cap D$ , and  $x \in D$ , with  $x \neq y$ .

**Proof.** In the proof of Theorem 6.6 there is only one point where the outer  $X$ -ball condition is used. Consider  $y \in B_d(P, \frac{1}{2}\epsilon) \cap D$  and assume that  $d(y, \partial D) \leq d(x, y)$ . Choose  $2r = \frac{d(y, \partial D)}{a}$  and observe that if  $z \in B(y, r)$  then  $d(z, y) < ar \leq \epsilon/2$ . Consequently  $d(z, P) \leq d(z, y) + d(y, P) \leq \epsilon$ , and we can apply Theorem 6.9 to the function  $G(x, z)$  concluding the proof in the same way as before.  $\square$

**Corollary 6.11.** Let  $D \subset \mathbb{R}^n$  be a  $C^\infty$  domain. If  $D$  satisfies the uniform outer  $X$ -ball condition in a neighborhood  $V$  of  $\Sigma$ , then for any  $x_o \in D$  and every open neighborhood  $U$  of  $\partial D$ , such that  $x_o \notin \overline{U}$ , one has  $\|G(x_o, \cdot)\|_{\mathcal{L}^{1,\infty}(U)} \leq C(x_o, D, V, U, X)$ .

**Proof.** Observe that  $D$  is regular for the Dirichlet problem. The regularity away from the characteristic set follows by Theorem 3.12 and the regularity in a neighborhood of  $\Sigma$  is a consequence of the uniform outer  $X$ -ball condition and of the cited results in [Ci],[D], [NS] and [CDG3]. Denote by  $V$  the neighborhood of  $\Sigma$  where the uniform outer  $X$ -ball condition holds.

In view of the compactness of  $\Sigma$ , we have that  $W = \bigcup_{P \in \Sigma} B(P, 2\epsilon) \subset V$ , for some  $\epsilon > 0$ . We will consider also the set  $A = \bigcup_{P \in \Sigma} B(P, \frac{1}{2}\epsilon) \subset W$ . In view of Theorem 3.12, we have that  $G(x_o, \cdot) \in C^\infty(\bar{D} \setminus \{A \cup \{x_o\}\})$ . In particular,  $G(x_o, \cdot)$  is smooth in  $U \setminus A$ . This implies the estimate  $\|G(x_o, \cdot)\|_{\mathcal{L}^{1,\infty}(U \setminus A)} \leq C_0 = C_0(x_o, D, V, X)$ . To complete the proof of the corollary we consider  $y \in A$  and observe that there must be a  $P \in \Sigma$  such that  $y \in B(P, \frac{1}{2}\epsilon)$ . Denote by  $Q$  the homogeneous dimension associated to the system  $X$  in a neighborhood of  $D$ . In view of Theorem 6.10 we have that  $|XG(x_o, y)| \leq Cd(y, x_o)^{1-Q} \leq C_1$ , with  $C_1$  depending only on  $X, D$  and  $U$ . At this point we choose  $C(x_o, D, V, U, X) = \min\{C_0, C_1\}$ , and the proof is concluded.  $\square$

## 7. The subelliptic Poisson kernel and a representation formula for $\mathcal{L}$ -harmonic functions

In this section we establish a basic Poisson type representation formula for smooth domains that satisfy the outer  $X$ -ball condition in a neighborhood of the characteristic set. This results generalizes an analogous representation formula for the Heisenberg group  $\mathbb{H}^n$  obtained by Lanconelli and Uguzzoni in [LU1] and extended in [CGN2] to groups of Heisenberg type. Consider a domain  $D$  which is regular for the Dirichlet problem. For a fixed point  $x_o \in D$  we respectively denote by  $\Gamma(x) = \Gamma(x, x_o)$  and  $G(x) = G(x, x_o)$ , the fundamental solution of  $\mathcal{L}$ , and the Green function for  $D$  and  $\mathcal{L}$  with pole at  $x_o$ . Recall that  $G(x) = \Gamma(x) - h(x)$ , where  $h$  is the unique  $\mathcal{L}$ -harmonic function with boundary values  $\Gamma$ . We also note that due to the assumption that  $D$  be regular,  $G, h$  are continuous in any relatively compact subdomain of  $\bar{D} \setminus \{x_o\}$ . We next consider a  $C^\infty$  domain  $\Omega \subset \bar{\Omega} \subset \bar{D}$  containing the point  $x_o$ . For any  $u, v \in C^\infty(\bar{D})$  we obtain from the divergence theorem

$$(7.1) \quad \int_{\Omega} [u \mathcal{L}v - v \mathcal{L}u] dx = \sum_{j=1}^m \int_{\partial\Omega} [v X_j u - u X_j v] \langle X_j, \nu \rangle d\sigma,$$

where  $\nu$  denotes the outer unit normal and  $d\sigma$  the surface measure on  $\partial\Omega$ . By Hörmander's hypoellipticity theorem [H] the function  $x \rightarrow \Gamma(x_o, x)$  is in  $C^\infty(D \setminus \{x_o\})$ . By Sard's theorem there exists a sequence  $s_k \nearrow \infty$  such that the sets  $\{x \in \mathbb{R}^n \mid \Gamma(x_o, x) = s_k\}$  are  $C^\infty$  manifolds. Since by (2.7) the fundamental solution has a singularity at  $x_o$ , we can assume without restriction that such manifolds are strictly contained in  $\Omega$ . Set  $\epsilon_k = F(x_o, s_k^{-1})$ , where  $F(x_o, \cdot)$  is the inverse function of  $E(x_o, \cdot)$  introduced in section two. The sets  $B(\epsilon_k) = B(x_o, \epsilon_k) \subset \bar{B}(x_o, \epsilon_k) \subset \Omega$  are a sequence of smooth  $X$ -balls shrinking to the point  $x_o$ . We note explicitly that the outer unit normal on  $\partial B(\epsilon_k)$  is  $\nu = -\frac{D\Gamma(\cdot, x_o)}{|D\Gamma(\cdot, x_o)|}$ .

Applying (7.1) with  $v(x) = G(x)$ , and  $\Omega$  replaced by  $\Omega_{\epsilon_k} = \Omega \setminus \bar{B}(\epsilon_k)$ , where one has  $\mathcal{L}G = 0$ , we find

$$\begin{aligned} \int_{\Omega_{\epsilon_k}} G \mathcal{L}u dx &= \sum_{j=1}^m \int_{\partial\Omega} [u X_j G - G X_j u] \langle X_j, \nu \rangle d\sigma \\ &\quad + \sum_{j=1}^m \int_{\partial B(\epsilon_k)} [G X_j u - u X_j G] \langle X_j, \nu \rangle d\sigma. \end{aligned}$$

Again the divergence theorem gives

$$(7.2) \quad \int_{B(\epsilon_k)} \mathcal{L}u dx = - \sum_{j=1}^m \int_{\partial B(\epsilon_k)} X_j u \langle X_j, \nu \rangle d\sigma.$$

Using (7.2), and the fact that  $G = \Gamma - h$ , we find

$$\begin{aligned}
 (7.3) \quad & \sum_{j=1}^m \int_{\partial B(\epsilon_k)} [G X_j u - u X_j G] \langle X_j, \nu \rangle d\sigma \\
 &= \frac{1}{E(x_o, \epsilon_k)} \sum_{j=1}^m \int_{\partial B(\epsilon_k)} X_j u \langle X_j, \nu \rangle d\sigma - \sum_{j=1}^m \int_{\partial B(\epsilon_k)} h X_j u \langle X_j, \nu \rangle d\sigma \\
 &\quad - \sum_{j=1}^m \int_{\partial B(\epsilon_k)} u X_j \Gamma \langle X_j, \nu \rangle d\sigma + \sum_{j=1}^m \int_{\partial B(\epsilon_k)} u X_j h \langle X_j, \nu \rangle d\sigma \\
 &= - \frac{1}{E(x_o, \epsilon_k)} \int_{B(\epsilon_k)} \mathcal{L}u dx + \int_{\partial B(\epsilon_k)} u \frac{|X\Gamma|^2}{|D\Gamma|} d\sigma \\
 &\quad + \sum_{j=1}^m \int_{\partial B(\epsilon_k)} u X_j h \langle X_j, \nu \rangle d\sigma - \sum_{j=1}^m \int_{\partial B(\epsilon_k)} h X_j u \langle X_j, \nu \rangle d\sigma .
 \end{aligned}$$

Using (5.3) we find

$$\int_{\partial B(\epsilon_k)} u \frac{|X\Gamma|^2}{|D\Gamma|} d\sigma = u(x_o) - \int_{B(\epsilon_k)} \mathcal{L}u \left[ \Gamma - \frac{1}{E(x_o, \epsilon_k)} \right] dx .$$

Keeping in mind that  $u, h \in C^\infty(\overline{\Omega})$ , from the estimates (2.7) and the fact that

$$\frac{|B(\epsilon_k)|}{E(x_o, \epsilon_k)} \leq C \epsilon_k^2,$$

letting  $k \rightarrow \infty$ , so that  $\epsilon_k \rightarrow 0$ , we conclude from (7.2), (7.3),

$$(7.4) \quad u(x_o) = \sum_{j=1}^m \int_{\partial\Omega} [G X_j u - u X_j G] \langle X_j, \nu \rangle d\sigma + \int_{\Omega} G \mathcal{L}u dx .$$

To summarize what we have found we introduce the following definition.

**Definition 7.1.** *Given a bounded open set  $\Omega \subset \overline{\Omega} \subset \mathbb{R}^n$  of class  $C^1$ , at every point  $y \in \partial\Omega$  we let*

$$\mathbf{N}^X(y) = (\langle \nu(y), X_1(y) \rangle, \dots, \langle \nu(y), X_m(y) \rangle) ,$$

where  $\nu(y)$  is the outer unit normal to  $\Omega$  in  $y$ . We also set

$$W(y) = |\mathbf{N}^X(y)| = \sqrt{\sum_{j=1}^m \langle \nu(y), X_j(y) \rangle^2} .$$

If  $y \in \partial\Omega \setminus \Sigma$ , we set

$$\nu^X(y) = \frac{\mathbf{N}^X(y)}{|\mathbf{N}^X(y)|} .$$

One has  $|\nu^X(y)| = 1$  for every  $y \in \partial D \setminus \Sigma$ .

We note explicitly from Definitions 3.11 and 7.1 that one has for the characteristic set  $\Sigma$  of  $\Omega$

$$\Sigma = \{y \in \partial\Omega \mid W(y) = 0\} .$$

Using the quantities introduced in this definition we can express (7.4) in the following suggestive way.

**Proposition 7.2.** *Let  $D \subset \mathbb{R}^n$  be a bounded open set with (positive) Green function  $G$  of the sub-Laplacian (1.2) and consider a  $C^2$  domain  $\Omega \subset \overline{\Omega} \subset D$ . For any  $u \in C^\infty(D)$  and every  $x \in \Omega$  one has*

$$\begin{aligned} u(x) &= \int_{\partial\Omega} G(x, y) \langle Xu(y), \mathbf{N}^X(y) \rangle d\sigma(y) - \int_{\partial\Omega} u(y) \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) \\ &\quad + \int_{\Omega} G(x, y) \mathcal{L}u(y) dy . \end{aligned}$$

If moreover  $\mathcal{L}u=0$  in  $D$ , then

$$u(x) = \int_{\partial\Omega} G(x, y) \langle Xu(y), \mathbf{N}^X(y) \rangle d\sigma(y) - \int_{\partial\Omega} u(y) \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) .$$

In particular, the latter equality gives for every  $x \in \Omega$

$$\int_{\partial\Omega} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) = -1 .$$

**Remark 7.3.** *If  $u \in C^\infty(\overline{D})$ , then we can weaken the hypothesis on  $\Omega$  and require only  $\overline{\Omega} \subset \overline{D}$  rather than  $\overline{\Omega} \subset D$ .*

We consider next a  $C^\infty$  domain  $D \subset \mathbb{R}^n$  satisfying the uniform outer  $X$ -ball condition in a neighborhood of  $\Sigma$ . Our purpose is to pass from the interior representation formula in Proposition 7.2 to one on the boundary of  $\partial D$ . The presence of characteristic points becomes important now. The following result due to Derriidj [De1, Theorem 1] will be important in the sequel.

**Theorem 7.4.** *Let  $D \subset \mathbb{R}^n$  be a  $C^\infty$  domain. If  $\Sigma$  denotes its characteristic set, then  $\sigma(\Sigma) = 0$ .*

We now define two functions on  $D \times (\partial D \setminus \Sigma)$  which play a central role in the results of this paper. They constitutes subelliptic versions of the Poisson kernel from classical potential theory. The former function  $P(x, y)$  is the Poisson kernel for  $D$  and the sub-Laplacian (1.2) with respect to surface measure  $\sigma$ . The latter  $K(x, y)$  is instead the Poisson kernel with respect to the perimeter measure  $\sigma_X$ . This comment will be clear after we prove Theorem 7.10 below.

**Definition 7.5** (Subelliptic Poisson kernels). *With the notation of Definition 7.1, for every  $(x, y) \in D \times (\partial D \setminus \Sigma)$  we let*

$$(7.5) \quad P(x, y) = - \langle XG(x, y), \mathbf{N}^X(y) \rangle .$$

We also define

$$(7.6) \quad K(x, y) = \frac{P(x, y)}{W(y)} = - \langle XG(x, y), \boldsymbol{\nu}^X(y) \rangle .$$

We extend the definition of  $P$  and  $K$  to all  $D \times \partial D$  by letting  $P(x, y) = K(x, y) = 0$  for any  $x \in D$  and  $y \in \Sigma$ . According to Theorem 7.4 the extended functions coincide  $\sigma$ -a.e. with the kernels in (7.5), (7.6).

It is important to note that if we fix  $x \in D$ , then in view of Theorem 3.12 the functions  $y \rightarrow P(x, y)$  and  $y \rightarrow K(x, y)$  are  $C^\infty$  up to  $\partial D \setminus \Sigma$ . The following estimates, which follow immediately from (7.5) and (7.6), will play an important role in the sequel. For  $(x, y) \in D \times (\partial D \setminus \Sigma)$  we have

$$(7.7) \quad P(x, y) \leq W(y) |XG(x, y)| , \quad K(x, y) \leq |XG(x, y)| .$$

We now introduce a new measure on  $\partial D$  by letting

$$(7.8) \quad d\sigma_X = W d\sigma .$$

We observe that since we are assuming that  $D \in C^\infty$  the density  $W$  is smooth and bounded on  $\partial D$  and therefore (7.8) implies that  $d\sigma_X \ll d\sigma$ . In view of this observation Theorem 7.4 implies that also  $\sigma_X(\Sigma) = 0$ .

**Remark 7.6.** We mention explicitly that the measure  $d\sigma_X$  in (7.8) is the  $X$ -perimeter measure  $P_X(D; \cdot)$  (following De Giorgi) concentrated on  $\partial D$ . To explain this point we recall that for any open set  $\Omega \subset \mathbb{R}^n$

$$(7.9) \quad P_X(D; \Omega) = \text{Var}_X(\chi_D; \Omega) ,$$

where  $\text{Var}_X$  indicates the sub-Riemannian  $X$ -variation introduced in [CDG2] and also developed in [GN1]. Given a bounded  $C^2$  domain  $D \subset \mathbb{R}^n$  one obtains from [CDG2] that

$$(7.10) \quad P_X(D; \Omega) = \int_{\partial D \cap \Omega} W \, d\sigma .$$

From (7.10) one concludes that for every  $y \in \partial D$  and every  $r > 0$

$$(7.11) \quad \sigma_X(\partial D \cap B_d(y, r)) = P_X(D; B_d(y, r)) ,$$

which explains the remark. The measure  $\sigma_X = P_X(D; \cdot)$  on  $\partial D$  plays a pervasive role in the analysis and geometry of sub-Riemannian spaces, and its intrinsic properties have many deep implications both in subelliptic pde's and in geometric measure theory. For an account of some of these aspects we refer the reader to [DGN2].

**Proposition 7.7.** Let  $D \subset \mathbb{R}^n$  be a bounded  $C^\infty$  domain satisfying the uniform outer  $X$ -ball condition in a neighborhood of its characteristic set  $\Sigma$ . For every  $x \in D$  we have

$$\int_{\partial D} P(x, y) d\sigma(y) = 1 = \int_{\partial D} K(x, y) d\sigma_X(y) .$$

**Proof.** We fix  $x \in D$  and recall that  $\Sigma$  is a compact set. In view of Theorem 7.4 we can choose an exhaustion of  $D$  with a family of  $C^\infty$  connected open sets  $\Omega_k \subset \overline{\Omega}_k \subset \overline{D}$ , with  $\Omega_k \nearrow D$  as  $k \rightarrow \infty$ , such that  $\partial\Omega_k = \Gamma_k^1 \cup \Gamma_k^2$ , with  $\Gamma_k^1 \subset \partial D \setminus \Sigma$ ,  $\Gamma_k^1 \nearrow \partial D$ ,  $\sigma(\Gamma_k^2) \rightarrow 0$ . By Proposition 7.2 (and the remark following it) we obtain for every  $k \in \mathbb{N}$

$$(7.12) \quad \begin{aligned} -1 &= \int_{\partial\Omega_k} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) \\ &= \int_{\partial\Gamma_k^1} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) + \int_{\partial\Gamma_k^2} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) . \end{aligned}$$

We now pass to the limit as  $k \rightarrow \infty$  in the above integrals. Using Corollary 6.11 and  $\sigma(\Gamma_k^2) \rightarrow 0$ , we infer that

$$\lim_{k \rightarrow \infty} \int_{\partial\Gamma_k^2} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) = 0 .$$

Theorem 3.12, Corollary 6.11, and the fact that  $\Gamma_k^1 \nearrow \partial D$ , allow to use dominated convergence and obtain

$$\lim_{k \rightarrow \infty} \int_{\partial\Gamma_k^1} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) = \int_{\partial D} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) .$$

In conclusion, we have found

$$-1 = \int_{\partial D} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) ,$$

which, in view of (7.5), proves the first identity. To establish the second identity we return to (7.12), which in view of (7.6), (7.8) we can rewrite as follows

$$\begin{aligned} 1 &= - \int_{\partial\Gamma_k^1} \langle XG(x, y), \boldsymbol{\nu}^X(y) \rangle d\sigma_X(y) - \int_{\partial\Gamma_k^2} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) \\ &= \int_{\partial\Gamma_k^1} K(x, y) d\sigma_X(y) - \int_{\partial\Gamma_k^2} \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) . \end{aligned}$$

Since as we have observed  $d\sigma_X \ll d\sigma$ , in view of the second estimate  $K(x, y) \leq |XG(x, y)|$  in (7.7), we can again use Theorem 3.12, Corollary 6.11 and dominated convergence (with respect to  $\sigma_X$ ) to conclude that

$$\lim_{k \rightarrow \infty} \int_{\partial\Gamma_k^1} K(x, y) d\sigma_X(y) = \int_{\partial D} K(x, y) d\sigma_X(y) .$$

This completes the proof.  $\square$

**Theorem 7.8.** *Let  $D$  satisfy the assumptions in Proposition 7.7. If  $\phi \in C^\infty(\partial D)$  assumes a single constant value in a neighborhood of  $\Sigma$ , then  $H_\phi^D \in \mathcal{L}^{1,\infty}(D)$ . Furthermore, if for  $\phi \in C(\partial D)$  we have  $H_\phi^D \in \mathcal{L}^{1,\infty}(D)$ , then*

$$H_\phi^D(x) = \int_{\partial D} P(x, y) \phi(y) d\sigma(y) = \int_{\partial D} K(x, y) \phi(y) d\sigma_X(y) , \quad x \in D .$$

**Proof.** We start with the proof of the regularity result. Let  $\phi$  be as in the first part of the statement. We mention explicitly that, by definition,  $\phi$  is  $C^\infty$  in a neighborhood of  $\partial D$ . Denote by  $U$  a neighborhood of  $\Sigma$  in which the function  $\phi$  is constant and along which the domain  $D$  satisfies the uniform outer  $X$ -ball condition. As in the proof of Corollary 6.11, we can assume that  $U = \bigcup_{P \in \Sigma} B_d(P, \epsilon)$ , for some  $\epsilon = \epsilon(U, X) > 0$ . If we denote by  $R_0$  the constant involved in the definition of the uniform outer  $X$ -ball (see Definition 6.8), then we can always select a smaller constant so that  $\epsilon = 2aR_0$  (here  $a > 1$  is the constant from (2.8)). In view of Proposition 7.7 we can assume without loss of generality that  $\phi$  vanishes in a neighborhood of  $\Sigma$  and  $\max_{\partial D} |\phi| = 1$ .

We want to show that the horizontal gradient of  $H_\phi^D$  is in  $L^\infty$  in such neighborhood. By Theorem 3.12 the conclusion  $H_\phi^D \in \mathcal{L}^{1,\infty}(D)$  will follow. Fix  $x_o \in \Sigma$ , and  $0 < r < R_o$ , where  $R_o$  is as in Definition 6.2. Theorem 6.3 implies

$$(7.13) \quad |H_\phi^D(y)| \leq C \frac{d(y, x_o)}{r}$$

for every  $y \in D$ . Let now  $x \in B(x_o, r/2) \cap D$  and consider the metric ball  $B_d(x, a^{-1}\tau) \subset B(x, \tau)$ , see (2.8), where  $\tau = \frac{d(x, \partial D)}{4}$ . Corollary 5.3 implies

$$(7.14) \quad |XH_\phi^D(x)| \leq \frac{C}{d(x, \partial D)} H_\phi^D(x) .$$

Pick  $P \in \partial D$  such that  $d(x, P) = d(x, \partial D)$ . Observe that  $d(P, \Sigma) \leq d(P, x_o) \leq d(P, x) + d(x, x_o) \leq 2d(x, x_o) \leq aR_0 = \epsilon/2$ . In particular we can apply once more Theorem 6.3, and obtain (7.13) with  $P$  in place of  $x_o$ . Arguing in this way we find

$$H_\phi^D(x) \leq C \frac{d(x, P)}{r} = C \frac{d(x, \partial D)}{r} .$$

The latter inequality and (7.14) imply

$$|XH_\phi^D(x)| \leq \frac{C}{r} .$$

This proves that  $|XH_\phi^D| \in L^\infty(B(x_o, r/2) \cap D)$ . To establish the second part of the theorem, we take a function  $\phi \in C(\partial D)$  for which  $H_\phi^D \in \mathcal{L}^{1,\infty}(D)$ . We fix  $x \in D$  and consider the sequence of  $C^\infty$  domains  $\Omega_k$  as in the proof of Proposition 7.7. Proposition 7.2 gives

$$(7.15) \quad H_\phi^D(x) = \int_{\partial\Omega_k} G(x, y) \langle X(H_\phi^D)(y), \mathbf{N}^X(y) \rangle d\sigma(y) - \int_{\partial\Omega_k} H_\phi^D(y) \langle XG(x, y), \mathbf{N}^X(y) \rangle d\sigma(y) .$$

At this point the conclusion follows along the lines of the proof of Proposition 7.7.  $\square$

**Proposition 7.9.** *Let  $D$  be a  $C^\infty$  domain. i) If  $D$  satisfies the uniform outer  $X$ -ball condition in a neighborhood of  $\Sigma$ , then  $P(x, y) \geq 0$  and  $K(x, y) \geq 0$  for each  $(x, y) \in D \times \partial D$ ; ii) If  $D$  satisfies the uniform outer  $X$ -ball condition, then there exists a constant  $C_D > 0$  such that for  $(x, y) \in D \times \partial D$*

$$0 \leq P(x, y) \leq C_D W(y) \frac{d(x, y)}{|B_d(x, d(x, y))|}, \quad 0 \leq K(x, y) \leq C_D \frac{d(x, y)}{|B_d(x, d(x, y))|}.$$

*In particular, if we fix  $x \in D$ , then for any open set  $U$  containing  $\partial D$ , such that  $x \notin \overline{U}$ , one has  $K(x, \cdot) \in L^\infty(\overline{D} \cap U)$ .*

**Proof.** We start with the proof of part (i). We argue by contradiction. If for some  $x \in D$  and  $x_o \in \partial D$  we had  $P(x, x_o) = \alpha < 0$ , then  $x_o \notin \Sigma$ . By Theorem 3.12 there exists a sufficiently small  $r > 0$  such that  $P(x, x') \leq \alpha/2$  for every  $x' \in B(x_o, 2r) \cap \partial D$ . We can also assume that  $d(x_o, \Sigma) > 2r$ . We now choose  $\phi \in C^\infty(\partial D)$  such that  $0 \leq \phi \leq 1$ ,  $\phi \equiv 1$  on  $B(x_o, r) \cap \partial D$  and  $\phi \equiv 0$  outside  $B(x_o, 3r/2) \cap \partial D$ . Theorem 3.2 implies  $H_\phi^D \geq 0$  in  $D$ . By the Harnack inequality we must have  $H_\phi^D(x) > 0$ . On the other hand, Theorem 7.8 gives

$$H_\phi^D(x) \leq \frac{\alpha}{2} \int_{B(x_o, 3r/2) \cap \partial D} \phi(y) d\sigma(y) \leq 0,$$

which gives a contradiction. The proof of part (ii) is an immediate consequence of (7.7) and of Theorem 6.6. The estimate for  $K(x, y)$  follows from (7.6) and from the one for  $P(x, y)$ .  $\square$

We now fix  $x \in D$ . For every  $\sigma$ -measurable  $E \subset \partial D$  we set

$$\nu^x(E) = \int_E K(x, y) d\sigma_X(y).$$

According to Proposition 7.9,  $d\nu^x$  defines a Borel measure on  $\partial D$ . Using Theorems 7.4 and 7.8 we can now establish the main result of this section.

**Theorem 7.10.** *Let  $D \subset \mathbb{R}^n$  be a  $C^\infty$  domain possessing the uniform outer  $X$ -ball condition in a neighborhood of the characteristic set  $\Sigma$ . For every  $x \in D$ , we have  $\omega^x = \nu^x$ , i.e., for every  $\phi \in C(\partial D)$  one has*

$$H_\phi^D(x) = \int_{\partial D} \phi(y) K(x, y) d\sigma_X(y) = \int_{\partial D} \phi(y) P(x, y) d\sigma(y), \quad x \in D.$$

*In particular,  $d\omega^x$  is absolutely continuous with respect to  $d\sigma_X$  and  $d\sigma$ , and for every  $(x, y) \in D \times \partial D$  one has*

$$(7.16) \quad \frac{d\omega^x}{d\sigma_X}(y) = K(x, y), \quad \frac{d\omega^x}{d\sigma}(y) = P(x, y).$$

**Proof.** We begin with proving (7.16). Let  $F \subset \partial D$  be a Borel set. If  $F = \partial D$  then the result follows from Proposition 7.7. We now consider the case when the inclusion  $F \subset \partial D$  is strict. Choose  $\epsilon > 0$ . Since both  $K(x, y)$  and  $W(y)$  are bounded, there exists open sets  $E_\epsilon, F_\epsilon \subset \partial D$  such that  $F \subset F_\epsilon \subset \overline{F_\epsilon} \subset E_\epsilon$ , and  $\nu^x(E_\epsilon \setminus F) < \epsilon/2$ . Theorem 7.4 guarantees the existence of open sets  $\Sigma_\epsilon, U_\epsilon$  such that  $\Sigma \subset \Sigma_\epsilon \subset \overline{\Sigma_\epsilon} \subset U_\epsilon$  and  $\nu^x(U_\epsilon) < \epsilon/2$ . We now choose a function  $\phi \in C_o^\infty(\partial D)$  and  $0 \leq \phi \leq 1$  with  $\phi \equiv 1$  on  $U_\epsilon$  and  $\nu^x(\text{supp } \phi) < \frac{3}{4}\epsilon$ . We have

$$(7.17) \quad \begin{aligned} \omega^x(U_\epsilon) &= \int_{U_\epsilon} d\omega^x(y) \leq \int_{\partial D} \phi(y) d\omega^x(y) = H_\phi^D(x) \\ &(\text{by Theorem 7.8}) = \int_{\partial D} \phi(y) K(x, y) d\sigma_X(y) \leq \nu^x(\text{supp } \phi) < \frac{3}{4}\epsilon. \end{aligned}$$

Let now  $\psi_o, \psi_1 \in C_o^\infty(\partial D)$  such that  $0 \leq \psi_o, \psi_1 \leq 1$  and

$$\begin{aligned}\psi_o &\equiv 1 \text{ in } \partial D \setminus U_\epsilon, & \psi_o &\equiv 0 \text{ in } \Sigma_\epsilon, \\ \psi_1 &\equiv 1 \text{ in } F, & \psi_1 &\equiv 0 \text{ in } \partial D \setminus E_\epsilon.\end{aligned}$$

One has

$$\begin{aligned}\omega^x(F) &\leq \omega^x(U_\epsilon) + \omega^x(F \setminus U_\epsilon) \quad (\text{by (7.17)}) \\ &\leq \frac{3}{4}\epsilon + \int_{\partial D} \psi_o(y)\psi_1(y) d\omega^x(y) \\ &= \frac{3}{4}\epsilon + H_{\psi_o\psi_1}^D(x) \quad (\text{by Theorem 7.8}) \\ &= \frac{3}{4}\epsilon + \int_{\partial D} \psi_o(y)\psi_1(y)K(x,y) d\sigma_X(y) \leq \frac{3}{4}\epsilon + \nu^x(E_\epsilon) \\ &= \frac{3}{4}\epsilon + \nu^x(F) + \nu^x(E_\epsilon \setminus F) < \nu^x(F) + \frac{7}{4}\epsilon.\end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, we infer that  $\omega^x(F) \leq \nu^x(F)$ . If we repeat the same argument with  $E_\epsilon \setminus F$  playing the role of the set  $F$ , we can prove  $\omega^x(E_\epsilon \setminus F) \leq \nu^x(E_\epsilon \setminus F)$ . This allows to exchange the role of  $\omega^x$  and  $\nu^x$  in the computations above and conclude  $\nu^x(F) \leq \omega^x(F)$ .

To complete the proof of the theorem we now use (7.16). From the definition of harmonic measure we know that for each  $\phi \in C(\partial D)$  and  $x \in D$  we have

$$(7.18) \quad H_\phi^D(x) = \int_{\partial D} \phi(y) d\omega^x(y).$$

On the other hand (7.16) yields  $d\omega^x(y) = K(x,y)d\sigma_X(y)$ . If we substitute the latter in (7.18) we reach the conclusion.  $\square$

## 8. Reverse Hölder inequalities for the Poisson kernel

This section is devoted to proving the main results of this paper, namely Theorems 1.3, 1.4, 1.5 and 1.6. In the course of the proofs we will need some basic results about  $NTA_X$  domains from the paper [CG1]. We begin by recalling the relevant definitions.

**Definition 8.1.** *We say that a connected, bounded open set  $D \subset \mathbb{R}^n$  is a non-tangentially accessible domain with respect to the system  $X = \{X_1, \dots, X_m\}$  ( $NTA_X$  domain, hereafter) if there exist  $M, r_o > 0$  for which:*

- (i) *(Interior corkscrew condition) For any  $x_o \in \partial D$  and  $r \leq r_o$  there exists  $A_r(x_o) \in D$  such that  $\frac{r}{M} < d(A_r(x_o), x_o) \leq r$  and  $d(A_r(x_o), \partial D) > \frac{r}{M}$ . (This implies that  $B_d(A_r(x_o), \frac{r}{2M})$  is  $(3M, X)$ -nontangential.)*
- (ii) *(Exterior corkscrew condition)  $D^c = \mathbb{R}^n \setminus D$  satisfies property (i).*
- (iii) *(Harnack chain condition) There exists  $C(M) > 0$  such that for any  $\epsilon > 0$  and  $x, y \in D$  such that  $d(x, \partial D) > \epsilon$ ,  $d(y, \partial D) > \epsilon$ , and  $d(x, y) < C\epsilon$ , there exists a Harnack chain joining  $x$  to  $y$  whose length depends on  $C$  but not on  $\epsilon$ .*

We note the following lemma which will prove useful in the sequel and which follows directly from Definition 8.1.

**Lemma 8.2.** *Let  $D \subset \mathbb{R}^n$  be  $NTA_X$  domain, then there exist constants  $C, R_1$  depending on the  $NTA_X$  parameters of  $D$  such that for every  $y \in \partial D$  and every  $0 < r < R_1$  one has*

$$C |B_d(y, r)| \leq \min\{|D \cap B_d(y, r)|, |D^c \cap B_d(y, r)|\} \leq C^{-1} |B_d(y, r)|.$$

*In particular, every  $NTA_X$  domain has positive density at every boundary point and therefore it is regular for the Dirichlet problem (see Definition 3.6, Proposition 3.7, and Theorem 3.5).*



In the sequel, for  $y \in \partial D$  and  $r > 0$  we denote by

$$\Delta(y, r) = \partial D \cap B_d(y, r)$$

the surface metric ball centered at  $y$  with radius  $r$ . We next prove a basic non-degeneracy property of the horizontal perimeter measure  $d\sigma_X$  in (7.8).

**Theorem 8.3.** *Let  $D \subset \mathbb{R}^n$  be a  $NTA_X$  domain of class  $C^2$ , then there exist  $C^*, R_1 > 0$  depending on  $D$ ,  $X$  and on the  $NTA_X$  parameters of  $D$  such that for every  $y \in \partial D$  and every  $0 < r < R_1$*

$$\sigma_X(\Delta(y, r)) \geq C^* \frac{|B_d(y, r)|}{r}.$$

*In particular,  $\sigma_X$  is lower 1-Ahlfors according to [DGN2] and  $\sigma_X(\Delta(y, r)) > 0$ .*

**Proof.** According to (I) in Theorem 1.15 in [GN1] every metric ball  $B_d(y, r)$  is a  $PS_X$  (Poincaré-Sobolev) domain with respect to the system  $X$ . We can thus apply the isoperimetric inequality Theorem 1.18 in [GN1] to infer the existence of  $R_1 > 0$  such that for every  $y \in \partial D$  and every  $0 < r < R_1$

$$\min\{|D \cap B_d(y, r)|, |D^c \cap B_d(y, r)|\}^{\frac{Q-1}{Q}} \leq C_{iso} \frac{\text{diam } B_d(y, r)}{|B_d(y, r)|^{\frac{1}{Q}}} P_X(D; B_d(y, r)),$$

where  $Q$  is the homogeneous dimension of a fixed bounded set  $U$  containing  $\overline{D}$ . On the other hand, every  $NTA_X$  domain is a  $PS_X$  domain. We can thus combine the latter inequality with (7.11) and Lemma 8.2 to finally obtain

$$\sigma_X(\Delta(y, r)) \geq C^* \frac{|B_d(y, r)|}{r}.$$

This proves the theorem. □

**Corollary 8.4.** *Let  $D \subset \mathbb{R}^n$  be a  $NTA_X$  domain of class  $C^2$  satisfying the upper 1-Ahlfors assumption in iv) of Definition 1.1. Then the measure  $\sigma_X$  is 1-Ahlfors, in the sense that there exist  $\tilde{A}, R_1 > 0$  depending on the  $NTA_X$  parameters of  $D$  and on  $A > 0$  in iv), such that for every  $y \in \partial D$ , and every  $0 < r < R_1$ , one has*

$$(8.1) \quad \tilde{A} \frac{|B_d(y, r)|}{r} \leq \sigma_X(\Delta(y, r)) \leq \tilde{A}^{-1} \frac{|B_d(y, r)|}{r}.$$

*In particular, the measure  $\sigma_X$  is doubling, i.e., there exists  $C > 0$  depending on  $\tilde{A}$  and on the constant  $C_1$  in (2.5), such that*

$$(8.2) \quad \sigma_X(\Delta(y, 2r)) \leq C \sigma_X(\Delta(y, r)).$$

*for every  $y \in \partial D$  and  $0 < r < R_1$ .*

**Proof.** According to Theorem 8.3 the measure  $\sigma_X$  is lower 1-Ahlfors. Since by iv) of Definition 1.1 it is also upper 1-Ahlfors, the conclusion (8.1) follows. From the latter and the doubling condition (2.5) for the metric balls, we reach the desired conclusion (8.2). □

The following results from [CG1] play a fundamental role in this paper.

**Theorem 8.5.** *Let  $D \subset \mathbb{R}^n$  be a  $NTA_X$  domain with relative parameters  $M, r_o$ . There exists a constant  $C > 0$ , depending only on  $X$  and on the  $NTA_X$  parameters of  $D$ ,  $M$  and  $r_o$ , such that for every  $x_o \in \partial D$  one has*

$$\omega^{A_r(x_o)}(\Delta(x_o, r)) \geq C.$$

**Theorem 8.6** (Doubling condition for  $\mathcal{L}$ -harmonic measure). *Consider a  $NTA_X$  domain  $D \subset \mathbb{R}^n$  with relative parameters  $M, r_o$ . Let  $x_o \in \partial D$  and  $r \leq r_o$ . There exist  $C > 0$ , depending on  $X, M$  and  $r_o$ , such that*

$$\omega^x(\Delta(x_o, 2r)) \leq C \omega^x(\Delta(x_o, r))$$

for any  $x \in D \setminus B_d(x_o, Mr)$ .

**Theorem 8.7** (Comparison theorem). *Let  $D \subset \mathbb{R}^n$  be a  $X$ -NTA domain with relative parameters  $M, r_o$ . Let  $x_o \in \partial D$  and  $0 < r < \frac{r_o}{M}$ . If  $u, v$  are  $\mathcal{L}$ -harmonic functions in  $D$ , which vanish continuously on  $\partial D \setminus \Delta(x_o, 2r)$ , then for every  $x \in D \setminus B_d(x_o, Mr)$  one has*

$$C \frac{u(A_r(x_o))}{v(A_r(x_o))} \leq \frac{u(x)}{v(x)} \leq C^{-1} \frac{u(A_r(x_o))}{v(A_r(x_o))}$$

for some constant  $C > 0$  depending only on  $X, M$  and  $r_o$ .

For any  $y \in \partial\Omega$  and  $\alpha > 0$  a nontangential region at  $y$  is defined by

$$\Gamma_\alpha(y) = \{x \in \Omega \mid d(x, y) \leq (1 + \alpha)d(x, \partial\Omega)\}.$$

Given a function  $u$  the  $\alpha$ -nontangential maximal function of  $u$  at  $y \in \partial D$  is defined by

$$N_\alpha(u)(y) = \sup_{x \in \Gamma_\alpha(y)} |u(x)|.$$

**Theorem 8.8.** *Let  $D \subset \mathbb{R}^n$  be a  $NTA_X$  domain. Given a point  $x_1 \in D$ , let  $f \in L^1(\partial D, d\omega^{x_1})$  and define*

$$u(x) = \int_{\partial D} f(y) d\omega^x(y), \quad x \in D.$$

Then,  $u$  is  $\mathcal{L}$ -harmonic in  $D$ , and:

- (i)  $N_\alpha(u)(y) \leq CM_{\omega^{x_1}}(f)(y)$ ,  $y \in \partial D$ ;
- (ii)  $u$  converges non-tangentially a.e. ( $d\omega^{x_1}$ ) to  $f$ .

Theorem 8.7 has the following important consequence.

**Theorem 8.9.** *Let  $D \subset \mathbb{R}^n$  be a  $ADP_X$  domain, and let  $K(\cdot, \cdot)$  be the Poisson Kernel defined in (7.6). There exists  $r_1 > 0$ , depending on  $M$  and  $r_o$ , and a constant  $C = C(X, M, r_o, R_o) > 0$ , such that given  $x_o \in \partial D$ , for every  $x \in D \setminus B_d(x_o, Mr)$  and every  $0 < r < r_1$  one can find  $E_{x_o, x, r} \subset \Delta(x_o, r)$ , with  $\sigma_X(E_{x_o, x, r}) = 0$ , for which*

$$K(x, y) \leq C K(A_r(x_o), y) \omega^x(\Delta(x_o, r))$$

for every  $y \in \Delta(x_o, r) \setminus E_{x_o, x, r}$ .

**Proof.** Let  $x_o \in \partial D$ . For each  $y \in \Delta(x_o, r)$  and  $0 < s < r/2$  set

$$u(x) = \omega^x(\Delta(y, s)), \quad v(x) = \omega^x(\Delta(x_o, r/2)).$$

The functions  $u$  and  $v$  are  $\mathcal{L}$ -harmonic in  $D$  and vanish continuously on  $\partial D \setminus \Delta(x_o, 2r)$ . Theorem 8.7 gives

$$(8.3) \quad \frac{\omega^x(\Delta(y, s))}{\omega^x(\Delta(x_o, r/2))} \leq C \frac{\omega^{A_r(x_o)}(\Delta(y, s))}{\omega^{A_r(x_o)}(\Delta(x_o, r/2))}$$

for every  $x \in D \setminus B(x_o, Mr)$ . Applying (8.3) we thus find

$$(8.4) \quad \frac{\omega^x(\Delta(y, s))}{\omega^x(\Delta(x_o, r/2))} \leq C \frac{\omega^{A_r(x_o)}(\Delta(y, s))}{\omega^{A_r(x_o)}(\Delta(x_o, r/2))}.$$

Upon dividing by  $\sigma_X(\Delta(y, s))$  in (8.4) (observe that in view of Theorem 8.3 the  $\sigma_X$  measure of any surface ball  $\Delta(y, s)$  is strictly positive), one concludes

$$(8.5) \quad \frac{\omega^x(\Delta(y, s))}{\sigma_X(\Delta(y, s))} \leq C \frac{\omega^{A_r(x_o)}(\Delta(y, s))}{\sigma_X(\Delta(y, s))} \frac{\omega^x(\Delta(x_o, r/2))}{\omega^{A_r(x_o)}(\Delta(x_o, r/2))} .$$

Using Theorem 8.5 in the right-hand side of (8.5) we conclude

$$(8.6) \quad \frac{\omega^x(\Delta(y, s))}{\sigma_X(\Delta(y, s))} \leq C \frac{\omega^{A_r(x_o)}(\Delta(y, s))}{\sigma_X(\Delta(y, s))} \omega^x(\Delta(x_o, r)) .$$

We now observe that (8.2) in Corollary 8.4 allows to obtain a Vitali covering theorem and differentiate the measure  $\omega^x$  with respect to the horizontal perimeter measure  $\sigma_X$ . This means that for  $\sigma_X$ -a.e.  $y \in \Delta(x_o, r)$  the limit  $\lim_{s \rightarrow 0} \frac{\omega^x(\Delta(y, s))}{\sigma_X(\Delta(y, s))}$  exists and equals  $\frac{d\omega^x}{d\sigma_X}(y)$ . This being said, passing to the limit as  $s \rightarrow 0^+$  in (8.6) we obtain for  $\sigma_X$ -a.e.  $y \in \Delta(x_o, r)$

$$\frac{d\omega^x}{d\sigma_X}(y) \leq C \frac{d\omega^{A_r(x_o)}}{d\sigma_X}(y) \omega^x(\Delta(x_o, r)) .$$

Since by (7.16) in Theorem 7.10 we know that  $\frac{d\omega^x}{d\sigma_X}(y) = K(x, y)$ ,  $\frac{d\omega^{A_r(x_o)}}{d\sigma_X}(y) = K(A_r(x_o), y)$ , we have reached the desired conclusion. We observe in passing that the exceptional set here depends on  $x$  and on  $A_r(x_o)$ , but this fact will be inconsequential to us since we plan to integrate with respect to  $\sigma_X$  the above inequality on the surface ball  $\Delta(x_o, r)$ . □

We now turn to the

**Proof of Theorem 1.3.** We fix  $p > 1$ ,  $x_o \in \partial D$  and  $x_1 \in D$ . Let  $R_1$  be the minimum of the constants appearing in Definitions 6.2, 8.1, and in Theorem 8.9. Moreover, the constant  $R_1$  should be chosen so small that  $d(x_1, x_o) > MR_1$ . Let  $0 < r < R_1$ . If  $A_r(x_o)$  is a corkscrew for  $x_o$ , then by the definition of a corkscrew, the triangle inequality and (2.3) it is easy to see that we have for all  $y \in \Delta(x_o, r)$

$$(8.7) \quad d(A_r(x_o), y) \sim Cr \quad \text{and} \quad |B_d(x_o, r)| \leq C|B_d(A_r(x_o), d(A_r(x_o), y))|.$$

Now we have

$$\begin{aligned}
& \left( \frac{1}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} K(x_1, y)^p d\sigma_X(y) \right)^{\frac{1}{p}} \quad (\text{by (7.16)}) \\
&= \left( \frac{1}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} K(x_1, y)^{p-1} d\omega^{x_1}(y) \right)^{\frac{1}{p}} \quad (\text{by Theorem 8.9}) \\
&\leq C \left( \frac{\omega^{x_1}(\Delta(x_o, r))^{p-1}}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} K(A_r(x_o), y)^{p-1} d\omega^{x_1}(y) \right)^{\frac{1}{p}} \quad (\text{by (7.7)}) \\
&\leq C \left( \frac{\omega^{x_1}(\Delta(x_o, r))^{p-1}}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} |XG(A_r(x_o), y)|^{p-1} d\omega^{x_1}(y) \right)^{\frac{1}{p}} \quad (\text{by Theorem 6.6}) \\
&\leq C \left( \frac{\omega^{x_1}(\Delta(x_o, r))^{p-1}}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} \left( \frac{d(A_r(x_o), y)}{|B_d(A_r(x_o), d(A_r(x_o), y))|} \right)^{p-1} d\omega^{x_1}(y) \right)^{\frac{1}{p}} \quad (\text{by (8.7)}) \\
&\leq C \left( \frac{\omega^{x_1}(\Delta(x_o, r))^{p-1}}{\sigma_X(\Delta(x_o, r))} \left( \frac{r}{|B_d(x_o, r)|} \right)^{p-1} \int_{\Delta(x_o, r)} d\omega^{x_1}(y) \right)^{\frac{1}{p}} \quad (\text{by iv) in Definition 1.1}) \\
&\leq \frac{C}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} d\omega^{x_1}(y) \quad (\text{by (7.16)}) \\
&= \frac{C}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} K(x_1, y) d\sigma_X(y) .
\end{aligned}$$

This concludes the proof of the reverse Hölder inequality. Regarding absolute continuity, we already know from (7.16) that  $d\omega^{x_1}$  is absolutely continuous with respect to  $d\sigma_X$ . To prove that  $d\sigma_X$  is absolutely continuous with respect to  $d\omega^{x_1}$  we only need to observe that the reverse Hölder inequality for  $K$  established above and the doubling property for  $d\sigma_X$  from (8.2) in Corollary 8.4 allow us to invoke Lemma 5 from [CF].  $\square$

We next establish a reverse Hölder inequality for the kernel  $P(x, y)$  defined in (7.5). The main thrust of this result is that, under a certain balanced-degeneracy assumption on the surface measure  $\sigma$  of  $\partial D$ , it implies the mutual absolute continuity of  $\mathcal{L}$ -harmonic measure and surface measure. Given the fact that, as we have explained in the introduction, surface measure is not the natural measure in the subelliptic Dirichlet problem, being able to isolate a condition which guarantees such mutual absolute continuity has some evident important consequences. To state the main result we modify the class of  $ADP_X$  domains in Definition 1.1. Specifically, we pose the following

**Definition 8.10.** *Given a system  $X = \{X_1, \dots, X_m\}$  of smooth vector fields satisfying (1.1), we say that a connected bounded open set  $D \subset \mathbb{R}^n$  is  $\sigma$ -admissible for the Dirichlet problem (1.3), or simply  $\sigma$ - $ADP_X$ , if:*

- i)  $D$  is of class  $C^\infty$ ;
- ii)  $D$  is non-tangentially accessible ( $NTA_X$ ) with respect to the Carnot-Carathéodory metric associated to the system  $\{X_1, \dots, X_m\}$  (see Definition 8.1);
- iii)  $D$  satisfies a uniform tangent outer  $X$ -ball condition (see Definition 6.2);
- iv) There exist  $B, R_o > 0$  depending on  $X$  and  $D$  such that for every  $x_o \in \partial D$  and  $0 < r < R_o$  one has

$$\left( \max_{y \in \Delta(x_o, r)} W(y) \right) \sigma(\Delta(x_o, r)) \leq B \frac{|B_d(x_o, r)|}{r} .$$

We note that Definitions 1.1 and 8.10 differ only in part *iv*). Also, (7.8) gives

$$\sigma_X(\Delta(x_o, r)) = \int_{\Delta(x_o, r)} W(y) d\sigma(y) \leq \left( \max_{y \in \Delta(x_o, r)} W(y) \right) \sigma(\Delta(x_o, r)) .$$

This observation shows that

$$\sigma - ADP_X \subset ADP_X .$$

The reason for which we have referred to the new assumption on  $\sigma$  as a balanced-degeneracy condition is that, as we have seen in the introduction the measure  $\sigma$  badly degenerates on the characteristic set  $\Sigma$ . On the other hand, the angle function  $W$  vanishes on  $\Sigma$ , thus balancing such degeneracy.

**Proof of Theorem 1.5.** The relevant reverse Hölder inequality for  $P(x_1, \cdot)$  is proved starting from the second identity  $d\omega^{x_1} = P(x_1, \cdot) d\sigma$  in (7.16) and then arguing in a similar fashion as in the proof of Theorem 1.3 but using the non-degeneracy estimate in *iv*) of Definition 8.10 instead of the upper 1-Ahlfors assumption in Definition 1.1. We leave the details to the interested reader.  $\square$

A consequence of Theorem 1.5 and of Theorem 8.6 is the following result. We stress that such result would be trivial if the surface balls would just be the ordinary Euclidean ones, but this is not the case here. Our surface balls  $\Delta(y, r)$  are the metric ones. Another comment is that away from the characteristic set the next result would be already contained in those in [MM].

**Theorem 8.11.** *Let  $D \subset \mathbb{R}^n$  be a  $\sigma - ADP_X$  domain. There exist  $C, R_1 > 0$  depending on the  $\sigma - ADP_X$  parameters of  $D$  such that for every  $y \in \partial D$  and  $0 < r < R_1$ ,*

$$\sigma(\Delta(y, 2r)) \leq C \sigma(\Delta(y, r)) .$$

**Proof.** Applying Theorem 1.5 with  $p = 2$ , we find

$$\begin{aligned} \frac{1}{\sigma(\Delta(x_o, r))} \int_{\Delta(x_o, r)} P(x_1, y)^2 d\sigma(y) &\leq \left( \frac{C}{\sigma(\Delta(x_o, r))} \int_{\Delta(x_o, r)} P(x_1, y) d\sigma(y) \right)^2 \\ &= C \left( \frac{\omega^{x_1}(\Delta(x_o, r))}{\sigma(\Delta(x_o, r))} \right)^2 . \end{aligned}$$

This gives

$$\begin{aligned} \sigma(\Delta(x_o, 2r)) &\leq C \frac{\omega^{x_1}(\Delta(x_o, 2r))^2}{\int_{\Delta(x_o, 2r)} P(x_1, y)^2 d\sigma(y)} \quad (\text{by Theorem 8.6}) \\ &\leq C \frac{\omega^{x_1}(\Delta(x_o, r))^2}{\int_{\Delta(x_o, r)} P(x_1, y)^2 d\sigma(y)} \leq C \frac{\left( \int_{\Delta(x_o, r)} P(x_1, y) d\sigma(y) \right)^2}{\int_{\Delta(x_o, r)} P(x_1, y)^2 d\sigma(y)} \\ &\leq C \frac{\left( \int_{\Delta(x_o, r)} P(x_1, y)^2 d\sigma(y) \right) \left( \int_{\Delta(x_o, r)} d\sigma(y) \right)}{\int_{\Delta(x_o, r)} P(x_1, y)^2 d\sigma(y)} = C \sigma(\Delta(x_o, r)) . \end{aligned}$$

$\square$

Our final goal in this section is to study the Dirichlet problem for sub-Laplacians when the boundary data are in  $L^p$  with respect to either the measure  $\sigma_X$  or the surface measure  $\sigma$ . We thus turn to the

**Proof of Theorem 1.4.** The first step in the proof consists of showing that functions  $f \in L^p(\partial D, d\sigma_X)$  are resolvable for the Dirichlet problem (1.3). In view of Theorem 3.3 it is enough

to show that  $f \in L^1(\partial D, d\omega^{x_1})$  for some fixed  $x_1 \in D$ . This follows from (7.16) and Proposition 7.9, based on the following estimates

$$\begin{aligned} \int_{\partial D} |f(y)| d\omega^{x_1}(y) &= \int_{\partial D} |f(y)| K(x_1, y) d\sigma_X(y) \\ &\leq \left( \int_{\partial D} |f(y)|^p d\sigma_X(y) \right)^{\frac{1}{p}} \left( \int_{\partial D} K(x_1, y)^{p'} d\sigma_X(y) \right)^{\frac{1}{p'}} \\ &\leq C \left( \int_{\partial D} |f(y)|^p d\sigma_X(y) \right)^{\frac{1}{p}}. \end{aligned}$$

This shows that  $L^p(\partial D, d\sigma_X) \subset L^1(\partial D, d\omega^{x_1})$  and therefore, in view of Theorem 3.3, for each  $f \in L^p(\partial D, d\sigma_X)$  the generalized solution  $H_f^D$  exists and it is represented by

$$H_f^D(x) = \int_{\partial D} f(y) d\omega^x(y).$$

At this point we invoke Theorem 8.8 and obtain for every  $y \in \partial D$

$$(8.8) \quad N_\alpha(H_f^D)(y) \leq C M_{\omega^{x_1}}(f)(y).$$

Moreover,  $H_f^D$  converges non-tangentially  $d\omega^{x_1}$ -a.e. to  $f$ . By virtue of Theorems 1.3 and 1.5, we also have that  $H_f^D$  converges  $d\sigma_X$ -a.e. to  $f$ . To conclude the proof, we need to show that there exists a constant  $C$  depending on  $1 < p < \infty$ ,  $D$  and  $X$  such that

$$\|N_\alpha(H_f^D)\|_{L^p(\partial D, d\sigma_X)} \leq C \|f\|_{L^p(\partial D, d\sigma_X)},$$

for every  $f \in L^p(\partial D, d\sigma_X)$ . In order to accomplish this we start by proving the following intermediate estimate

$$(8.9) \quad \|M_{\omega^{x_1}}(f)\|_{L^p(\partial D, d\sigma_X)} \leq C \|f\|_{L^p(\partial D, d\sigma_X)}, \quad 1 < p \leq \infty.$$

Since  $p > 1$ , choose  $\beta$  so that  $0 < \beta < p$  and fix  $x_1 \in D$  as in Theorem 1.3. From (7.16) and the reverse Hölder inequality in Theorem 1.3 we have

$$\begin{aligned} &\frac{1}{\omega^{x_1}(\Delta(x_o, r))} \int_{\Delta(x_o, r)} f(y) d\omega^{x_1}(y) \\ &\leq \frac{1}{\omega^{x_1}(\Delta(x_o, r))} \left( \int_{\Delta(x_o, r)} |f(y)|^\beta d\sigma_X(y) \right)^{\frac{1}{\beta}} \left( \int_{\Delta(x_o, r)} K(x_1, y)^{\beta'} d\sigma_X(y) \right)^{\frac{1}{\beta'}} \\ &\leq C \frac{\sigma_X(\Delta(x_o, r))^{\frac{1}{\beta'}}}{\omega^{x_1}(\Delta(x_o, r))} \left( \frac{1}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} K(x_1, y) d\sigma_X(y) \right) \|f\|_{L^\beta(\Delta(x_o, r), d\sigma_X)} \\ &= C \left( \frac{1}{\sigma_X(\Delta(x_o, r))} \int_{\Delta(x_o, r)} |f(y)|^\beta d\sigma_X(y) \right)^{\frac{1}{\beta}}. \end{aligned}$$

If we now fix  $y \in \partial D$  and take the supremum on both sides of the latter inequality by integrating on every surface ball  $\Delta(x_o, r)$  containing  $y$ , we obtain

$$(8.10) \quad M_{\omega^{x_1}}(f)(y) \leq C M_{\sigma_X}(|f|^\beta)(y)^{\frac{1}{\beta}}.$$

By the doubling condition (8.2) in Corollary 8.4 we know that the space  $(\partial D, d(x, y), d\sigma_X)$  is a space of homogeneous type. This allows us to use the results in [CW] and invoke the continuity

in  $L^p(\partial D, d\sigma_X)$  of the Hardy-Littlewood maximal function obtaining

$$\begin{aligned} \|M_{\omega^{x_1}} f\|_{L^p(\partial D, d\sigma_X)}^p &\leq C \|M_{\sigma_X}(|f|^\beta)^{\frac{1}{\beta}}\|_{L^p(\partial D, d\sigma_X)}^p \\ &= \int_{\partial D} M_{\sigma_X}(|f|^\beta)^{\frac{p}{\beta}} d\sigma_X \leq C \int_{\partial D} |f|^p d\sigma_X = C \|f\|_{L^p(\partial D, d\sigma_X)}^p, \end{aligned}$$

which proves (8.9). The conclusion of the theorem follows at once from (8.8) and (8.9).  $\square$

Finally, we give the

**Proof of Theorem 1.6.** If the domain  $D$  is a  $\sigma$ - $ADP_X$ -domain, instead of a  $ADP_X$ -domain, then using Theorem 1.5 instead of Theorem 1.3 we can establish the solvability of the Dirichlet problem for boundary data in  $L^p$  with respect to the standard surface measure. Since the proof of the following result is similar to that of Theorem 1.4 (except that one needs to use the second identity  $d\omega^{x_1} = P(x_1, \cdot) d\sigma$  in (7.16) and also Theorem 8.11), we leave the details to the interested reader.  $\square$

## 9. A survey of examples and some open problems

In the study of boundary value problems for sub-Laplacians one faces two type of difficulties. On one side there is the elusive nature of the underlying sub-Riemannian geometry which makes most of the classical results hard to establish. On the other hand, any new result requires a detailed analysis of geometrically significant examples, without which the result itself would be devoid of meaning. This task is not easy, the difficulties being mostly related to the presence of characteristic points. In this perspective it becomes important to provide examples of  $ADP_X$ -domains. In this section we recall some of the pertinent results from recent literature.

**Examples of  $NTA_X$  domains.** In the classical setting Lipschitz and even  $BMO_1$  domains are  $NTA$  domains [JK]. In a Carnot-Carathéodory space it is considerably harder to produce examples of such domains, due to the presence of characteristic points on the boundary. In [CG1] it was proved that in a Carnot group of step two every  $C^{1,1}$  domain with cylindrical symmetry at characteristic points is  $NTA_X$ . In particular, the pseudo-balls in the natural gauge of such groups are  $NTA_X$ . This result was subsequently generalized by Monti and Morbidelli [MM].

**Theorem 9.1.** *In a Carnot group of step  $r = 2$  every bounded (Euclidean)  $C^{1,1}$  domain is  $NTA_X$  with respect to the Carnot-Carathéodory metric associated to a system  $X$  of generators of the Lie algebra.*

**Examples of domains satisfying the uniform outer  $X$ -ball property.** The following result provides a general class of domains satisfying the uniform  $X$ -ball condition, see [LU1] and [CGN2]. We recall the following definition from [CGN2]. Given a Carnot group  $\mathbb{G}$ , with Lie algebra  $\mathfrak{g}$ , a set  $A \subset \mathbb{G}$  is called *convex*, if  $\exp^{-1}(A)$  is a convex subset of  $\mathfrak{g}$ .

**Theorem 9.2.** *Let  $\mathbb{G}$  be a step two Carnot group of Heisenberg type with a given orthogonal set  $X = \{X_1, \dots, X_m\}$  of generators of its Lie algebra, and let  $D \subset \mathbb{G}$  be a convex set. For every  $R > 0$  and  $x_o \in \partial D$  there exists a  $X$ -ball  $B(x_o, R)$  such that (6.4) is satisfied. From this it follows that every bounded convex subset of  $\mathbb{G}$  satisfies the uniform outer  $X$ -ball condition. In particular, this is true for the gauge balls.*

We mention explicitly that, thanks to the results in [K], in every group of Heisenberg type with an orthogonal system  $X$  of generators of  $\mathfrak{g} = V_1 \oplus V_2$ , the fundamental solution of the sub-Laplacian associated with  $X$  is given by

$$\Gamma(x, y) = \frac{C(\mathbb{G})}{N(x^{-1}y)^{Q-2}},$$

where  $Q = \dim(V_1) + 2 \dim(V_2)$  is the homogeneous dimension of  $\mathbb{G}$ , and

$$N(x, y) = (|x|^4 + 16|y|^2)^{1/4},$$

is the non-isotropic Kaplan's gauge. Kaplan's formula for the fundamental solution shows, in particular, that in a group of Heisenberg type the  $X$ -balls coincide with the gauge pseudo-balls (incidentally, in this setting the gauge defines an actual distance, see [Cy]). As a consequence of this fact, the exterior  $X$ -balls in Theorem 9.2 can be constructed explicitly by finding the coordinates of their center through the solution of a linear system and a second order equation.

**Ahlfors type estimates for the perimeter measure.** Recall that if  $D \subset \mathbb{R}^n$  is a standard  $C^1$ , or even a Lipschitz domain, then there exist positive constants  $\alpha, \beta$  and  $R_o$  depending only on  $n$  and on the Lipschitz character of  $D$ , such that for every  $x_o \in \partial D$ , and every  $0 < r < R_o$  one has

$$(9.1) \quad \alpha r^{n-1} \leq \sigma(\partial D \cap B(x_o, r)) = P(D; B(x_o, r)) \leq \beta r^{n-1}.$$

Here, we have denoted by  $P(D, B(x_o, r))$  the perimeter of  $D$  in  $B(x_o, r)$  according to De Giorgi. Estimates such as (9.1) are referred to as the 1-Ahlfors property of surface measure. They play a pervasive role in Euclidean analysis especially in connection with geometric measure theory and its applications to the study of boundary value problems. In what follows we recall some basic regularity results for the  $X$ -perimeter measure which generalize (9.1) and play a central role in the applications of our results. We have mentioned in the introduction that from the standpoint of the Carnot-Carathéodory geometry, Euclidean smoothness of a domain is of no significance. Even for  $C^\infty$  domains one should not, therefore, expect 1-Ahlfors regularity in general, see [CG2] for various examples. For this reason we introduce the notion of type of a boundary point, and recall a result showing that if a domain possesses such property, then the corresponding  $X$ -perimeter satisfies Ahlfors regularity properties with respect to the metric balls.

Given a system of  $C^\infty$  vector fields  $X = \{X_1, \dots, X_m\}$  satisfying (1.1), consider a bounded  $C^1$  domain  $D \subset \mathbb{R}^n$  with an outer normal  $\mathbf{N}$ . We say that a point  $x_o \in \partial D$  is of type  $\leq 2$  if either there exists  $j_o \in \{1, \dots, m\}$  such that  $\langle X_{j_o}(x_o), \mathbf{N}(x_o) \rangle \neq 0$  (i.e.,  $x_o$  is non-characteristic, see Definition 3.11), or there exist indices  $i_o, j_o \in \{1, \dots, m\}$  such that  $\langle [X_{i_o}, X_{j_o}](x_o), \mathbf{N}(x_o) \rangle \neq 0$ . We say that  $D$  is of type  $\leq 2$  if every point  $x_o \in \partial D$  is of type  $\leq 2$ . We stress that when the system has rank  $r \leq 2$ , then every  $C^1$  domain is automatically of type  $\leq 2$ . An important instance is given by a Carnot group of step  $r = 2$ . In such a group, every bounded  $C^1$  domain is of type  $\leq 2$ . The following theorem is from [CG1].

**Theorem 9.3.** *Consider a bounded  $C^{1,1}$  domain  $D \subset \mathbb{R}^n$ . For every point  $x_o \in \partial D$  of type  $\leq 2$  there exist  $A = A(D, x_o) > 0$  and  $R_o = R_o(D, x_o) > 0$ , depending continuously on  $x_o$ , such that for any  $0 < r < R_o$  one has*

$$(9.2) \quad \sigma_X(\Delta(x_o, r)) \leq \left( \max_{y \in \Delta(x_o, r)} W(y) \right) \sigma(\Delta(x_o, r)) \leq A \frac{|B_d(x_o, r)|}{r}.$$

*The same conclusion holds if  $\partial D$  is real analytic in a neighborhood of  $x_o$ , regardless of the type of  $x_o$ .*

*If  $D$  is a bounded  $C^2$  domain, then for every point  $x_o \in \partial D$  of type  $\leq 2$  there exist  $A = A(D, x_o) > 0$  and  $R_o = R_o(D, x_o) > 0$ , depending continuously on  $x_o$ , such that for any  $0 < r < R_o$ , one has*

$$(9.3) \quad \sigma_X(\Delta(x_o, r)) \geq A^{-1} \frac{|B_d(x_o, r)|}{r}.$$

We mention that in Carnot groups of step  $r = 2$  the upper 1-Ahlfors estimate (9.2) was first proved in [DGN1], whereas for vector fields of rank  $r = 2$  the lower estimate (9.3) was first established in [DGN2]. In the setting of Hörmander vector fields, upper Ahlfors estimates for



the surface measure  $\sigma$  away from the characteristic set were first established in [MM2]. As a consequence of Theorem 9.3 we obtain the following

**Corollary 9.4.** *Let  $X = \{X_1, \dots, X_m\}$  be a set of  $C^\infty$  vector fields in  $\mathbb{R}^n$  satisfying Hörmander's condition with rank two, i.e. such that*

$$\text{span}\{X_1, \dots, X_m, [X_1, X_2], \dots, [X_{m-1}, X_m]\} = \mathbb{R}^n,$$

*at every point. For every bounded  $C^{1,1}$  domain  $D \subset \mathbb{R}^n$  the horizontal perimeter measure  $\sigma_X$  is a 1-Ahlfors measure. Moreover the stronger estimate (9.2) holds.*

As a consequence of the results listed above we obtain the following theorem which provides a large class of domains satisfying the  $ADP_X$  or even the stronger  $\sigma - ADP_X$  property.

**Theorem 9.5.** *Let  $\mathbb{G}$  be a Carnot group of Heisenberg type and denote by  $X = \{X_1, \dots, X_m\}$  a set of generators of its Lie algebra. Every  $C^\infty$  convex bounded domain  $D \subset \mathbb{G}$  is a  $ADP_X$  and also a  $\sigma - ADP_X$  domain. In particular, the gauge balls in  $\mathbb{G}$  are  $ADP_X$  and also  $\sigma - ADP_X$  domains.*

To conclude our review of Ahlfors type estimates, we bring up an interesting connection between 1-Ahlfors regularity of the  $X$ -perimeter  $\sigma_X$  and the Dirichlet problem for the sub-Laplacian, see [CG2]:

**Theorem 9.6.** *Let  $D$  be a bounded domain in a Carnot group  $\mathbb{G}$ . If the perimeter measure  $\sigma_X$  is 1-Ahlfors regular, then every  $x_o \in \partial D$  is regular for the Dirichlet problem.*

This result, in conjunction with a class of examples for non-regular domain constructed in [HH] yields the following

**Corollary 9.7.** *If  $r \geq 3$  and  $m_1 \geq 3$ , or  $m_1 = 2$  and  $r \geq 4$ , then there exist Carnot groups  $\mathbb{G}$  of step  $r \in \mathbb{N}$ , with  $\dim V_1 = m_1$ , and bounded,  $C^\infty$  domains  $D \subset \mathbb{G}$ , whose perimeter measure  $\sigma_X$  is not 1-Ahlfors regular.*

**Beyond Heisenberg type groups.** The above overview shows that, so far, the known examples of  $ADP_X$  domains are relative to group of Heisenberg type. What happens beyond such groups? For instance, what can be said even for general Carnot groups of step two? One of the difficulties here is to find examples of domains satisfying the outer tangent  $X$ -ball condition. The explicit construction in Theorem 9.2 above rests on the special structure of a group of Heisenberg type, and an extension to more general groups appears difficult due to the fact that, in a general group, the  $X$ -balls are not explicitly known and they may be quite different from the gauge balls. In this connection it would be desirable to replace the uniform outer  $X$ -ball condition with a *uniform outer gauge pseudo-ball condition* (clearly the two definitions agree for groups of Heisenberg type). It would be quite interesting to know whether for general Carnot groups a uniform outer gauge pseudo-ball condition would suffice to establish the boundedness of the horizontal gradient of the Green function near the characteristic set (this question is open even for Carnot groups of step two which are not of Heisenberg type!). Concerning the question of examples we have the following special results.

**Definition 9.8.** *Let  $\mathbb{G}$  be a Carnot group and denote by  $\mathfrak{g}$  its Lie algebra. We say that a family  $\mathcal{F}$  of smooth open subsets of  $\mathfrak{g}$  is a  $T$ -family if it satisfies*

- (i) *For any  $F \in \mathcal{F}$ , the manifold  $\partial F$  is diffeomorphic to the unit sphere in the Lie algebra.*
- (ii) *The family  $\mathcal{F}$  is left-invariant, i.e. for any  $x \in \mathbb{G}$  and  $F \in \mathcal{F}$  we have  $\log(x \exp(F)) \in \mathcal{F}$ .*

If  $D \subset \mathfrak{g}$  is a smooth subset and  $\mathcal{F}$  is a  $T$ -family, then we say that  $D$  is *tangent to  $\mathcal{F}$*  if for every  $x \in \partial D$  there exists  $F \in \mathcal{F}$  such that  $x \in \partial F$  and the tangent hyperplanes to  $\partial F$  and  $\partial D$  at  $x$  are identical, i.e.  $T_x \partial F = T_x \partial D$ .

**Theorem 9.9.** *Let  $\mathfrak{g}$  be the Lie algebra of a Carnot group of odd dimension. If  $D \subset \mathfrak{g}$  is a smooth open set and  $\mathcal{F}$  is a  $T$ -family, then  $D$  is tangent to  $\mathcal{F}$ .*

**Proof.** In order to avoid using  $\exp$  and  $\log$  maps for all  $x, y \in \mathfrak{g}$  we will denote by  $xy$  the algebra element  $\log(\exp x \exp y)$ . We will assume that  $\mathfrak{g}$  is endowed with a Euclidean metric, so that notions of orthogonality and projections can be used. Fix  $x_o \in \partial D$  and choose any element  $F \in \mathcal{F}$ . We will show that there exists  $z \in \mathfrak{g}$  such that the left-translation  $zF$  is tangent to  $\partial D$  at  $x_o$ .

Let  $n = \dim(\mathbb{G}) = \dim(\mathfrak{g})$  be odd, and denote by  $S^{n-1}$  the unit (Euclidean) sphere of dimension  $n - 1$ . Define the map  $N : \partial F \rightarrow S^{n-1}$  as follows: For each point  $x \in \partial F$  set  $\tilde{D} = xx_o^{-1}D$  and observe that this is a smooth open set with  $x \in \partial \tilde{D} \cap \partial F$ . Set

$N(x) =$  the outer unit normal to the boundary of the translated set  $\partial \tilde{D}$  at the point  $x$ .

This amounts to left-translating the point  $x_o$  to the point  $x$  and considering the unit normal to the translated domain at that point. The smoothness of  $D$  and of the group structure of  $\mathbb{G}$  implies that  $N$  is a smooth vector field in  $\partial F$ . In order to prove the theorem we need to show that for some point  $x \in \partial F$  the vector  $N(x)$  is orthogonal to  $T_x \partial F$ . In fact in that case the set  $F$  would be tangent to the translated set  $\tilde{D}$  at the point  $x_o$ , and its left translation  $x_o x^{-1}F$  could be chosen as the element of  $\mathcal{F}$  tangent to  $D$  at the point  $x_o$ . Recall that left translation, being a diffeomorphism, preserves the property of being tangent. The conclusion comes from the fact that there cannot be any smooth tangent non vanishing vector field on  $\partial F$  since it is diffeomorphic to  $S^{n-1}$ . Consequently the vector fields obtained by projecting  $N(x)$  on  $T_x \partial F$  must vanish for some point  $x \in \partial F$ . □

**Corollary 9.10.** *Let  $\mathbb{G}$  be a Carnot group of step two with odd-dimensional Lie algebra  $\mathfrak{g}$  and  $D \subset \mathfrak{g}$  be a smooth convex subset. If  $\mathcal{F}$  is a  $T$ -family, composed of convex subsets, and invariant by the transformation  $x \rightarrow x^{-1}$  then for any  $x \in \partial D$  there exists  $F \in \mathcal{F}$  such that  $F \subset D^c$ , and  $x \in \partial F$ .*

**Proof.** In a Carnot group of step two the left translation map is affine and hence preserves convexity. The same holds for the inverse map. Consequently at any boundary point  $x_o \in \partial D$  there will be a convex manifold  $F \in \mathcal{F}$  tangent to  $D$  at  $x_o$ . Being  $D$  convex as well then  $D$  and  $F$  must either be on the same side or lay at different sides of the common tangent plane  $T_{x_o} \partial D$ . By translating  $x_o$  to the origin and considering either  $F$  or  $F^{-1}$  we can pick the manifold lying on the opposite side of  $D$  and hence disjoint from it. □

Choosing appropriate  $T$ -families of convex sets we can now prove our two main results concerning the uniform outer gauge pseudo-ball and  $X$ -ball conditions.

**Corollary 9.11.** *Let  $\mathbb{G}$  be a Carnot group of step two with odd-dimensional Lie algebra  $\mathfrak{g}$ . Given a convex set  $D \subset \mathbb{G}$ , for every  $x_o \in \partial D$  and every  $r > 0$  there exists a gauge pseudo-ball  $B(x_1, r)$  which is tangent to  $\partial D$  in  $x_o$  from the outside, i.e., such that (6.4) is satisfied. Furthermore, every bounded convex set in  $\mathbb{G}$  satisfies the uniform outer gauge pseudo-ball condition.*

**Proof.** If  $D$  is smooth then the proof follows from the immediate observation that the gauge balls are convex sets in the Lie algebra and are diffeomorphic to  $S^{n-1}$  (see for instance [F2]). For non-smooth convex domains  $D$ , we consider  $x_o \in \partial D$  and a new domain  $\hat{D}$  obtained as the half space including  $D$  and with boundary  $T_{x_o} \partial D$ . Since  $\hat{D}$  is a smooth convex domain then we can apply to it the previous theorem and find an outer tangent gauge ball at the point  $x_o$  with radius  $r > 0$ . Clearly this ball will also be tangent to the original domain  $D$  at  $x_o$ , and will be contained entirely in the complement of  $D$ . □

**Corollary 9.12.** *Let  $\mathbb{G}$  be a Carnot group of step two with odd-dimensional Lie algebra  $\mathfrak{g}$ . If for every  $x \in \mathbb{R}^n$  and for  $r$  sufficiently small the  $X$ -balls  $B(x, r)$  are convex, and  $B(x^{-1}, r) = B(x, r)^{-1}$  then every bounded convex set in  $\mathbb{G}$  satisfies the uniform outer  $X$ -ball condition.*

**Proof.** We need only to show that the family of balls  $B(x, r)$  form a  $T$ -family. In [DG2] it is shown that  $X$ -balls are starlike with respect to the family of homogeneous dilations in the Carnot group. In particular, one has the estimate

$$\langle \nabla \Gamma(\cdot, x), Z \rangle > 0$$

on  $\partial B(x, r)$  where we have denoted by  $Z$  the generator of the homogeneous dilations. This inequality, coupled with Hörmander's hypoellipticity result, implies that  $\partial B(x, r)$  is a smooth manifold, while the starlike property immediately implies that  $\partial B(x, r)$  is diffeomorphic to the unit ball.  $\square$

We recall from the classical paper of Folland [F2] that in a Carnot group the fundamental solution of the sub-Laplacian is a function  $\Gamma(x, y) = \Gamma(y^{-1}x)$  and  $\Gamma(x^{-1}) = \Gamma^t(x)$ , where  $\Gamma^t$  is the fundamental solution of the transpose of the sub-Laplacian  $\mathcal{L}$ . However, a sub-Laplacian on a Carnot group is self-adjoint, hence  $\mathcal{L}^* = -\mathcal{L}$  and  $\Gamma(x) = \Gamma(x^{-1})$ . Let us denote by  $\|\cdot\|$  the group gauge, if we assume that for all  $x, y \in \mathbb{G}$  one has  $\Gamma(xy^{-1}) = \Gamma(yx^{-1})$  (this happens for instance if  $\Gamma(x) = \Gamma(\|x\|)$ ), and set  $B(x, r) = \{y \mid \Gamma(y^{-1}x) > c\}$  then

$$B(x, r)^{-1} = \{y^{-1} \mid \Gamma(x^{-1}y) > c\} = B(x^{-1}, r).$$

We conclude by explicitly noting that a serious obstruction to extending the previous results to Carnot groups of higher step consists in the fact that, unlike in the step two case, the group left-translation may not preserve the convexity of the sets.

**Beyond linear equations.** Another interesting direction of investigation for the subelliptic Dirichlet problem is provided by the study of solutions to the nonlinear equations which arise in connection with the case  $p \neq 2$  of the Folland-Stein Sobolev embedding. In this direction a first step has been recently taken in [GNg] where, among other results, Theorem 6.4 has been extended to the Green function of the nonlinear equation

$$(9.4) \quad \mathcal{L}_p u = \sum_{j=1}^{2n} X_j (|Xu|^{p-2} X_j u) = 0,$$

in the Heisenberg group  $\mathbb{H}^n$ . Here is the relevant result.

**Theorem 9.13.** *Let  $D \subset \mathbb{H}^n$  be a bounded domain satisfying the uniform outer  $X$ -ball condition. Given  $1 < p \leq Q$ , let  $G_{D,p}$  denote the Green function associated with (9.4) and  $D$ . Denote by  $g = (z, t), g' = (z', t') \in \mathbb{H}^n$ .*

(i) *If  $1 < p < Q$  there exists a constant  $C = C(\mathbb{G}, D, p) > 0$  such that*

$$G_{D,p}(g', g) \leq C \left( \frac{d(g, g')}{|B(g, d(g, g'))|} \right)^{1/(p-1)} d(g', \partial D), \quad g, g' \in D, \quad g' \neq g.$$

(ii) *If  $p = Q$ , then there exists  $C = C(\mathbb{G}, D) > 0$  such that*

$$G_{D,p}(g', g) \leq C \log \left( \frac{\text{diam}(D)}{d(g, g')} \right) \frac{d(g', \partial D)}{d(g, g')}, \quad g, g' \in D, \quad g' \neq g.$$

One might naturally wonder about results such as Theorem 6.6 in this setting. However, before addressing this question one has to understand the fundamental open question of the interior local bounds of the horizontal gradient of a solution to (9.4). For recent progress in this direction see the paper [MZZ].

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